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HAUSDORFF DIMENSION OF UNIFORMLY NON FLAT SETS WITH TOPOLOGY

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Abstract

Let d be an integer, and let E be a nonempty closed subset of \mathbb{R}^n . Assume that E is locally uniformly non flat, in the sense that for $x \in E$ and $r > 0$ small, $E \cap B(x, r)$ never stays $\varepsilon_0 r$ -close to an affine d -plane. Also suppose that E satisfies locally uniformly some appropriate d -dimensional topological nondegeneracy condition, like Semmes' Condition B. Then the Hausdorff dimension of E is strictly larger than d . We see this as an application of uniform rectifiability results on Almgren quasiminimal (restricted) sets.

1. Introduction

Let E be a closed subset in Euclidean space \mathbb{R}^n , and let an integer dimension $d < n$ be given. Suppose that at all scales and locations, E is both uniformly non flat (relative to that dimension) and satisfies some d -dimensional topological nondegeneracy condition. We want to show that the Hausdorff dimension of E is (strictly) larger than d .

A simple instance of this is when $d = 1$, E is connected, and its P. Jones numbers are such that $\beta_E(x, r) \geq \varepsilon$ for $x \in E$ and $0 < r \leq 1$. Then C. J. Bishop and P. W. Jones [BJ2] showed that E has Hausdorff dimension at least d^* , where $d^* > 1$ depends only on n and ε .

Here we shall keep the same condition to measure the uniform non-flatness of E . Call \mathcal{P} the set of affine planes of dimension d and set

$$(1.1) \quad \beta_E(x, r) = \inf_{P \in \mathcal{P}} \left\{ \sup \{ r^{-1} \text{dist}(y, P); y \in E \cap B(x, r) \} \right\}$$

for $x \in \mathbb{R}^n$ and $r > 0$. We require the existence of $\varepsilon_0 > 0$ and $r_0 > 0$ such that

$$(1.2) \quad \beta_E(x, r) \geq \varepsilon_0 \text{ for all } x \in E \text{ and } 0 < r \leq r_0.$$

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Note that using L^p variants of $\beta_E(x, r)$ would only make (1.2) harder to check, and would thus lead to weaker theorems below. We could also prove the results below with a bilateral version of $\beta_E(x, r)$. That is, we could replace (1.2) with the weaker requirement that for every choice of $x \in E$ and $0 < r \leq r_0$, we cannot find a plane P such that $\text{dist}(y, P) \leq \varepsilon_0 r$ for every $y \in E \cap B(x, r)$ and $\text{dist}(z, E) \leq \varepsilon_0 r$ for every $z \in P \cap B(x, r)$. The proof would be the same (just replace the weak geometric lemma by its bilateral version near (6.10)). But this remark is essentially useless, because with our topological nondegeneracy condition, the weaker analogue of (1.2) implies (1.2) (with a smaller ε_0).

For the topological nondegeneracy condition, there seems to be no obvious best choice, so we shall present two main ones and a few consequences, and hope that they will turn out to be the right ones. The general idea, as in [Se2], [DS2], and Section 12 of [DS3], is to require some property of topological nature (i.e., for instance, invariant under deformations) that would automatically imply that E is at least d -dimensional. Connectedness (in dimension $d = 1$), as in the result of Bishop and Jones, would be an example. We want to require something like this at all scales and locations, too.

For our first topological condition, we shall consider deformations of E in balls $B(x, r)$. These deformations will be given by one-parameter families of Lipschitz mappings $\{\varphi_t\}$, $0 \leq t \leq 1$, defined on \mathbb{R}^n , with the following properties:

$$(1.3) \quad \varphi_t(\overline{B}(x, r)) \subset \overline{B}(x, r) \text{ for each } t \in [0, 1];$$

$$(1.4) \quad \text{for each } y \in \mathbb{R}^n, \varphi_t(y) \text{ is a continuous function of } t \in [0, 1];$$

$$(1.5) \quad \varphi_t(y) = y \text{ for } t = 0 \text{ and for } y \in \mathbb{R}^n \setminus B(x, r);$$

$$(1.6) \quad \text{dist}(\varphi_t(y), E) \leq \alpha_0 r \text{ for } t \in [0, 1] \text{ and } y \in E \cap B(x, r),$$

where the constant $\alpha_0 \in (0, 1)$ is given in advance. Thus we only allow ourselves to move $E \cap B(x, r)$ in a thin tube around E . This is important, because it may force $\varphi_1(E)$ to keep some of the topological properties of E .

Minor modifications of this are possible. For instance, we may require that $t \rightarrow \varphi_t$ be a continuous function of t , valued in Lipschitz functions. We could also have decided to define φ_t only on $[E \cap B(x, r)] \cup \partial B(x, r)$, but it would then have been easy to extend it to \mathbb{R}^n . Note that in (1.3) we do not require any bound on the Lipschitz constant of φ_t , and probably we could even replace the Lipschitz requirement with simple continuity.

For our first topological condition, we shall require the existence of two small positive constants η_0 and δ_0 such that

$$(1.7) \quad H^d[B(x, (1 - \eta_0)r) \cap \varphi_1(E)] \geq \delta_0 r^d$$

when (1.3)–(1.6) hold. Let us already give the corresponding statement.

Theorem 1.8. *For each choice of positive constants ε_0 , r_0 , α_0 , η_0 , and δ_0 , we can find $d^* > d$ such that $\mathcal{H}\dim(E) \geq d^*$ whenever E is a nonempty closed set in \mathbb{R}^n such that (1.2) holds and*

$$(1.9) \quad (1.7) \text{ holds for all choices of } x \in E, 0 < r \leq r_0, \text{ and of Lipschitz deformations } \{\varphi_t\} \text{ with the properties (1.3), (1.4), (1.5), and (1.6).}$$

Here $\mathcal{H}\dim(E)$ denotes the Hausdorff dimension of E . Thus our conclusion is that the Hausdorff measure $H^m(E)$ is infinite for every $m < d^*$. See for instance [Fa] for details about $\mathcal{H}\dim$ and H^d .

We shall give in Section 2 a few additional comments about our topological condition (1.7), and in particular check that it is implied by S. Semmes' Condition B (a uniform separation property in codimension 1). However it is a little unpleasant that in (1.7) we require a lower bound on $H^d[B(x, (1 - \eta_0)r) \cap \varphi_1(E)]$, rather than the more natural $H^d(\varphi_1(E \cap B(x, r)))$. Since we fear that the difference may cause trouble in potential applications, we shall give a slightly more general, but also somewhat more technical, version of Theorem 1.8, where we use approximations of E by small d -dimensional skeletons. See Theorem 3.8 below.

Both theorems are easy to localize. That is, if E satisfies the hypotheses of these theorems for all balls $B(x, r)$ contained in an open set Ω , then $E \cap B(y, \rho)$ also has Hausdorff dimension at least d^* when $y \in E \cap \Omega$ and $\rho > 0$. This will be clear from the proof.

Our proof will be a little indirect; the general idea is the following. We shall proceed by contradiction, assume that E is not too large, and construct a functional and a minimizer F for this functional. By definition of the functional and the topological condition, F will have a big intersection with a small modification E_ρ of E . We shall also check that F is a quasiminimal set (in the terminology of F. J. Almgren [A1], a restricted set), then use the local uniform rectifiability estimates from [DS3] to show that F is very flat in most balls; a contradiction with (1.2) will ensue. All this will be in the spirit of Section 12 of [DS3]. See the beginning of Section 4 for a slightly more precise description of the argument.

Even though the author likes the indirect argument below, he is aware that there could be much simpler, direct proofs of Theorems 1.8 and 3.8.

Potential applications for the results of this paper could be lower bounds on the dimension of limit sets of some groups, as was the case with the original paper of Bishop and Jones. But the author knows very little about the subject.

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2. Condition B and other topological nondegeneracy conditions

The following condition was introduced by S. Semmes [Se1], in connection with L^2 -boundedness of singular integral operators on subsets of \mathbb{R}^n . This is a topological condition of codimension 1.

Definition 2.1. Let E be a closed set in \mathbb{R}^n . We say that E satisfies Condition B locally if there are constants $\alpha_1 > 0$ and $r_0 > 0$ such that, for all choices of $x \in E$ and $0 < r \leq r_0$, we can find two balls B_1 and B_2 contained in $B(x, r) \setminus E$, of radius $\alpha_1 r$, and such that B_1 and B_2 lie in different connected components of $B(x, r) \setminus E$.

We added “locally” because in [Se1] and related works, the property is assumed to hold for all $r < \text{diam}(E)$. Also, we weakened the property a little, because normally one requires the balls B_1 and B_2 to be contained in different connected components of $\mathbb{R}^n \setminus E$.

Lemma 2.2. *If the nonempty closed set E satisfies Condition B locally, then it satisfies the hypothesis (1.9) in Theorem 1.8, with $d = n - 1$. Hence, if E also satisfies (1.2) with $d = n - 1$, then $\mathcal{H}\dim(E) \geq d^* > n - 1$, for some d^* that depends only on n , d , α_1 , and ε_0 .*

So let E satisfy Condition B locally, and let us prove (1.9); the rest of the lemma will follow from Theorem 1.8. We shall keep the same r_0 and take

$$(2.3) \quad \alpha_0 = \eta_0 = \alpha_1/2 \text{ and } \delta_0 = 2^{-n+1} c_{n-1} \alpha_1^{n-1},$$

where c_{n-1} denotes the Lebesgue measure of the unit ball in \mathbb{R}^{n-1} .

Let $x \in E$ and $0 < r \leq r_0$ be given, and let us apply Condition B. We get two balls $B_1 = B(x_1, \alpha_1 r)$ and $B_2 = B(x_2, \alpha_1 r)$, with the properties stated in Definition 2.1. Set $B'_j = B(x_j, \alpha_1 r/2)$ for $j = 1, 2$.

Next let $\{\varphi_t\}$ be a family of Lipschitz functions, with the properties (1.3)–(1.6). Note that for $y \in E$ and $0 \leq t \leq 1$, $(\varphi_t(y), E) \leq \alpha_0 r =$

$\alpha_1 r/2$, by (1.6) or (1.5). Thus

$$(2.4) \quad \varphi_t(E) \text{ never meets } B'_1 \cup B'_2.$$

Note that $\varphi_t(\partial B(x, r)) = \partial B(x, r)$ does not meet $B'_1 \cup B'_2$ either, by (1.5) and because $B_j \subset B(x, r)$. We claim that

$$(2.5) \quad B'_1 \text{ and } B'_2 \text{ lie in different connected components of } \mathbb{R}^n \setminus \varphi_1(E \cup \partial B(x, r)).$$

This comes directly from 4.3 on Chapter XVII of [Du, p. 360], (for instance). The point is that if $y_1 \in B'_1$ and $y_2 \in B'_2$, then y_1 and y_2 lie in different components of $\mathbb{R}^n \setminus [E \cup \partial B(x, r)]$ by definition of B_1 and B_2 , and this stays the same with $\varphi_t(E)$ for all t , because $\varphi_t(E)$ never passes through y_1 or y_2 . The proof uses Borsuk's theorem on extensions of maps into spheres, and Brouwer's theorem that says that the identity map on the sphere is not nullhomotopic.

Call L the line through the centers x_1 and x_2 , and π the orthogonal projection onto the hyperplane P through x_1 and perpendicular to L . Let us check that

$$(2.6) \quad \pi[\varphi_1(E) \cap B(x, r - \eta_0 r)] \text{ contains } P \cap B'_1.$$

Let $\xi \in P \cap B'_1$ be given, and call L_ξ the line through ξ parallel to L . Then L_ξ meets B'_1 and B'_2 , because $\pi(B'_1) = \pi(B'_2) = P \cap B'_1$. Since B'_1 and B'_2 lie in different components of $\mathbb{R}^n \setminus \varphi_1(E \cup \partial B(x, r))$, we can find a point z_ξ in $\varphi_1(E \cup \partial B(x, r)) \cap L_\xi$, somewhere between the intersections of L_ξ with B'_1 and B'_2 . Note that z_ξ lies in the convex hull of $B'_1 \cup B'_2$, which itself is contained in $B(x, r - \alpha_1 r/2) = B(x, r - \eta_0 r)$. So $z_\xi \in \varphi_1(E) \cap B(x, r - \eta_0 r)$, and since $\pi(z_\xi) = \xi$, we see that $\xi \in \pi[\varphi_1(E) \cap B(x, r - \eta_0 r)]$, as needed for (2.6).

Since π is Lipschitz with constant 1, (2.6) says that

$$(2.7) \quad \begin{aligned} H^d(\varphi_1(E) \cap B(x, r - \eta_0 r)) &\geq H^d(\pi[\varphi_1(E) \cap B(x, r - \eta_0 r)]) \\ &\geq H^d(P \cap B(x_1, \alpha_1 r/2)) \\ &= 2^{-n+1} c_{n-1} \alpha_1^{n-1} r^d, \end{aligned}$$

which is exactly (1.7) with $\delta_0 = 2^{-n+1} c_{n-1} \alpha_1^{n-1}$.

This completes our proof of (1.9), and Lemma 2.2 follows. \square

It is reasonably easy to come up with generalizations of Condition B in higher codimensions, but it is a little harder to guess which one is the most natural. We shall give such a condition here, more as an example of what can be done than a final answer to a natural problem. The following definition is a local version of Definition 3 on [Da, p. 106].

Definition 2.8. Let E be a closed set in \mathbb{R}^n . We say that $E \in SS(d, n)$ locally if there are constants $\alpha_1 > 0$ and $r_0 > 0$ such that, for all choices of $x \in E$ and $0 < r \leq r_0$, we can find an affine subspace W of dimension $n - d$ and an Euclidean sphere $\Sigma \subset W \cap B(x, r/2)$ of dimension $n - d - 1$, such that

$$(2.9) \quad \text{dist}(\Sigma, E) \geq 2\alpha_1 r$$

and Σ is linked with E in $B(x, r)$, in the sense that there is no one-parameter family $\{f_t\}$ of continuous functions on \mathbb{R}^n such that

$$(2.10) \quad f_t(\overline{B}(x, r)) \subset \overline{B}(x, r) \text{ for each } t \in [0, 1],$$

$$(2.11) \quad \text{for each } y \in \mathbb{R}^n, f_t(y) \text{ is a continuous function of } t \in [0, 1],$$

$$(2.12) \quad f_t(y) = y \text{ for } t = 0 \text{ and for } y \in \mathbb{R}^n \setminus B(x, r),$$

$$(2.13) \quad \text{dist}(f_t(y), \Sigma) \geq \alpha_1 r \text{ for } t \in [0, 1] \text{ and } y \in E \cap B(x, r),$$

and

$$(2.14) \quad f_1(E \cap B(x, r)) \subset \partial B(x, r).$$

In other words, there is no way to deform $E \cap B(x, r)$ inside of $\overline{B}(x, r)$ so that it lands on $\partial B(x, r)$, without crossing a small neighborhood of Σ .

When $d = n - 1$, Σ is composed of two points, and our condition looks a lot like Condition B, except that instead of saying that the two centers of the B_j in Condition B lie in different components of $\mathbb{R}^n \setminus E$, we say that we cannot deform E past the two points without crossing them. But here the conditions are equivalent, by Borsuk's and Brouwer's theorems.

In higher codimensions, one could come up with other ways to say that E is linked with Σ , and it is not clear that they are all equivalent. The linking condition in Definition 2.8 is probably not the most natural one (as the case of codimension 1 suggests), but at least we can use it easily. Also, there are probably a few algebraic ways to make sure that E and Σ are linked and $E \in SS(d, n)$ locally, but this is a slightly different subject. See Remark 2.24 for a little more about this.

Lemma 2.15. *If E is a closed set and $E \in SS(d, n)$ locally, then we can find α_0 , δ_0 , and η_0 such that (1.7) holds for all choices of $x \in E$, $0 < r \leq r_0$, and all families $\{\varphi_t\}$ of Lipschitz deformations such that (1.3), (1.4), (1.5), and (1.6) hold.*

In other words, (1.9) holds. The constants α_0 , δ_0 , and η_0 depend only on the constants in $SS(d, n)$; in fact, we can keep the same r_0 and take

$$(2.16) \quad \alpha_0 = \alpha_1, \eta_0 = 1/2, \text{ and } \delta_0 = 2^{-d} c_d \alpha_1^d.$$

Now Theorem 2.9 says that if E is closed, nonempty, and $E \in SS(d, n)$ and (1.2) holds, then the Hausdorff dimension of E is at least d^* , where $d^* > d$ depends only on n , d , α_1 , and ε_0 .

Let us prove the lemma. Choose constants as in (2.16). As before, we start with $x \in E$ and $0 < r \leq r_0$, Definition 2.8 gives a sphere Σ , and we want to use it to prove (1.7) for all families $\{\varphi_t\}$ of Lipschitz functions that satisfy (1.3)–(1.6).

So we give ourselves such a family, assume that (1.7) fails, and try to reach a contradiction. The idea will be to use $\{\varphi_t\}$ to find deformations $\{f_t\}$ that satisfy (2.10)–(2.14).

Note that $\{\varphi_t\}$ automatically satisfies (2.10)–(2.12), by (1.3)–(1.5). Also,

$$(2.17) \quad \text{dist}(\varphi_t(E), \Sigma) \geq \text{dist}(E, \Sigma) - \alpha_0 r \geq \alpha_0 r$$

by (1.6), (2.9), and because $\alpha_0 = \alpha_1$. In particular, (2.13) holds for $\{\varphi_t\}$. But φ_1 does not necessarily satisfy (2.14); we only know that (1.7) fails, i.e., that

$$(2.18) \quad H^d[\varphi_1(E) \cap B(x, r/2)] < \delta_0 r^d = 2^{-d} c_d \alpha_1^d r^d,$$

by (2.16). We have to deform $\varphi_1(E)$ some more, to send it to $\partial B(x, r)$.

Call x_0 the center of Σ , W^\perp the affine d -plane through x_0 perpendicular to W , and π the orthogonal projection onto W^\perp . Then (2.18) says that

$$(2.19) \quad H^d[\varphi_1(E) \cap B(x, r/2)] < H^d(W^\perp \cap B(x_0, \alpha_1 r/2)).$$

Hence $\pi(\varphi_1(E) \cap B(x, r/2))$ does not contain $W^\perp \cap B(x_0, \alpha_1 r/2)$.

Let $z \in W^\perp \cap B(x_0, \alpha_1 r/2)$ be such that $D = \pi^{-1}(z) \cap B(x, r/2)$ does not meet $\varphi_1(E)$. Set $U = \{y \in B(x, r); \text{dist}(y, \Sigma) < \alpha_1 r\}$, and call V the convex hull of U . Thus V is an $\alpha_1 r$ -neighborhood of the ball of W enclosed in Σ .

Also set $D_\varepsilon = \{y \in V; \text{dist}(y, D) < \varepsilon\}$. Since $\varphi_1(E)$ is closed and does not meet D , we can choose $\varepsilon < \alpha_1 r/2$ so small that D_ε does not meet $\varphi_1(E)$.

Now we can find a continuous deformation $\{\psi_t\}$, $0 \leq t \leq 1$, such that $\psi_0(y) = y$ everywhere, $\psi_t(y) = y$ when $y \in \mathbb{R}^n \setminus V$,

$$(2.20) \quad \psi_t(\overline{V} \setminus [U \cup D_\varepsilon]) \subset \overline{V} \setminus [U \cup D_\varepsilon] \text{ for all } t,$$

and

$$(2.21) \quad \psi_1(V \setminus [U \cup D_\varepsilon]) \subset \partial V.$$

Let us just give an idea of the construction. We first define ψ_1 , and then we shall take $\psi_t(y) = ty + (1-t)\psi_1(y)$. We keep $\psi_1(y) = y$ on D . For $y \in V \setminus D_\varepsilon$, we call y^* the point of D such that $y^* - y$ is parallel to W^\perp , and then let $\psi_1(y)$ be the point of the half line from y^* through y that lies in ∂V . In other words, we move y straight away from D and parallel to W^\perp until we hit ∂V . For the rest of V , i.e., on $V \cap D_\varepsilon$, we interpolate nicely between the two definitions. The properties (2.20) and (2.21) are easy to check; the main point for (2.20) is that if $y \in V \setminus [U \cup D_\varepsilon]$ then all the points of the half-line from y^* through y that lie after y also lie out of $U \cup D_\varepsilon$.

We complete our deformation $\{\psi_t\}$ by another continuous family $\{h_t\}$, $0 \leq t \leq 1$, such that $h_t(y) = y$ for $t = 0$ and for $y \in \mathbb{R}^n \setminus B(x, r)$, $h_t(\overline{B}(x, r)) \subset \overline{B}(x, r)$ for all t ,

$$(2.22) \quad h_t(B(x, r) \setminus V) \subset \overline{B}(x, r) \setminus V \text{ for all } t,$$

$$\text{and } h_1(\overline{B}(x, r) \setminus V) \subset \partial B(x, r).$$

This one is very easy to construct, since V is essentially a smaller ball inside $B(x, r)$; we just need to push the points out to $\partial B(x, r)$.

Our family $\{f_t\}$ is obtained by composing the three families above. That is, we set $f_t(y) = \varphi_{3t}(y)$ for $0 \leq t \leq 1/3$, then $f_t(y) = \psi_{3t-1}(\varphi_1(y))$ for $1/3 < t \leq 2/3$, and finally $f_t(y) = h_{3t-2}(\psi_1(\varphi_1(y)))$ for $t > 2/3$. The properties (2.10)–(2.12) are straightforward. Let us check (2.13).

When $t \leq 1/3$, (2.13) follows from (2.17). We also get that $\varphi_1(E) \subset \mathbb{R}^n \setminus [U \cup D_\varepsilon]$, by definition of U and D_ε . Then (2.20) says that $f_t(E) \subset \mathbb{R}^n \setminus [U \cup D_\varepsilon]$ for $1/3 < t \leq 2/3$, which proves (2.13) in that case. Also note that

$$(2.23) \quad f_{2/3}(E) = \psi_1(\varphi_1(E)) \subset \mathbb{R}^n \setminus V,$$

by (2.21). Finally, for $t > 2/3$, $f_t(E) = h_{3t-2}(\psi_1(\varphi_1(E)))$ stays out of V by (2.22), and hence (2.13) holds.

Our last condition (2.14) holds because $f_1(E \cap B(x, r)) = h_1(\psi_1(\varphi_1(E \cap B(x, r)))) \subset h_1(\overline{B}(x, r) \setminus V) \subset \partial B(x, r)$, by (2.23) and (2.22).

By Definition 2.8, such a family $\{f_t\}$ cannot exist; this gives the desired contradiction and proves Lemma 2.15. \square

Remark 2.24. It would be easy to prove the analogue of Lemma 2.15 with a somewhat weaker definition of $SS(d, n)$. First, the Euclidean sphere Σ could be replaced with a bilipschitz image of such a sphere, for instance. In [Da] this was hard to do because one wanted to know that E has “big projection”, and for this one really wanted to show that the straight projection $\pi(E \cap B(x, r/2))$ contains $W^\perp \cap B(x_0, \alpha_1 r/2)$. The argument was the same as above (for $\varphi_t(y) \equiv y$); see below (2.19). Here we just want to prove (1.7) (or contradict (2.18)), and for this a bilipschitz image of a sphere is enough. Since the point is merely to get (1.7), one could probably replace spheres by other objects too.

In fact, we could get (1.7) from various other linking conditions, like the existence of a homotopically nontrivial mapping from $E \cap B(x, r)$ to a sphere S^d . We do not want to try to give a list here, because we would surely miss the most relevant example. But at least we should say that (1.7) has a stability property that should makes it pleasant to check. That is, if we choose δ_0 small enough, depending on n , d , and α_0 , then either (1.7) holds (for all deformations $\{\varphi_t\}$ in $B(x, t)$), or else we can even find $\{\varphi_t\}$ such that $B(x, (1 - \eta_0)r) \cap \varphi_1(E)$ has Hausdorff dimension at most $d - 1$. We can get this by a projection on skeletons, a little bit like when we will construct f and π in Section 5. The reader is kindly invited to consult Subsection 12.2 of [DS3] (called Stability of Sets) for this; more ideas on topological conditions that may lead to (1.7) can be found in Subsection 12.3 of [DS3] (Topological Interpretations).

3. Our main technical topological condition

For our main technical topological nondegeneracy condition, we need to introduce dyadic cubes and skeletons. Let $\rho > 0$ be a small number; the reader may think of ρ as a small power of 2, but since we often work at one scale at a time, this does not really matter. Our set of dyadic cubes of size ρ is

$$(3.1) \quad \mathcal{Q}(\rho) = \{Q = \rho[0, 1]^n + \rho\ell; \ell \in \mathbb{Z}^n\}.$$

For each $Q \in \mathcal{Q}(\rho)$ and each integer $m \in [0, n]$, we denote by $S_m(Q)$ the m -dimensional skeleton of Q . Thus $S_0(Q)$ is the set of all vertices of Q , $S_n(Q)$ is the cube Q itself, and in general $S_m(Q)$ is the union of all the m -dimensional faces of Q . It will be convenient to replace the closed set E with

$$(3.2) \quad E_\rho = \bigcup_{Q \in \mathcal{Q}(\rho); Q \cap E \neq \emptyset} S_d(Q),$$

where d is the same fixed integer as in the introduction. One of the points is that E_ρ has locally finite H^d -measure, while we expect E to be much larger. And the difference is not so large, because we can deform E onto a subset of E_ρ , at least if $\mathcal{H}\dim(E) < d + 1$. [See Lemma 3.14.]

Our second topological nondegeneracy condition *TND* is the following. We assume that for each choice of a big constant $C_1 > 0$, there are (small) positive constants r_0 , $\alpha < 1$, η , and δ_1 such for every choice of

$$(3.3) \quad x_1 \in E, \ 0 < r_1 \leq r_0, \text{ and } 0 < \rho \leq \frac{\alpha r_1}{2\sqrt{n}}$$

(the reader should not pay too much attention to the constant $2\sqrt{n}$ here, the point is merely to make sure that points of E_ρ lie within $\frac{\alpha r}{2}$ of E) such that

$$(3.4) \quad H^d(E_\rho \cap B(x_1, r_1)) \leq C_1 r_1^d,$$

there is a ball $B(x, r)$ centered on E and contained in $B(x_1, r_1)$ such that, for each one-parameter family $\{\varphi_t\}_{0 \leq t \leq 1}$ of Lipschitz functions on \mathbb{R}^n that satisfy (1.3), (1.4), (1.5), and for which

$$(3.5) \quad \text{dist}(\varphi_t(y), E) \leq \alpha r_1 \text{ for } t \in [0, 1] \text{ and } y \in E_\rho \cap B(x, r),$$

we have that

$$(3.6) \quad H^d(\varphi_1(E_\rho \cap B(x, r))) \geq \delta_1 r_1^d + H^d(E_\rho \cap A_{\eta r_1}(x, r)),$$

where we set

$$(3.7) \quad A_{\eta r_1}(x, r) = B(x, r) \setminus B(x, r - \eta r_1).$$

When this condition is satisfied, we shall also say that $E \in \textit{TND}$.

We still have an unpleasant boundary term in (3.6) (namely, $H^d(E_\rho \cap A_{\eta r_1}(x, r))$), but hopefully it will be easier to control in applications than the one in Theorem 1.8, in particular because we get some limited control from the extra condition (3.4).

Also, we get some extra flexibility by allowing our nondegeneracy condition (3.6) to hold only on some balls $B(x, r)$.

Note that we did not specify officially that $r \geq C^{-1}r_1$, but since we can apply (3.6) with $\varphi_t(y) \equiv y$, we get that $E_\rho \cap B(x, r) \not\subset A_{\eta r_1}(x, r)$, hence $r \geq \eta r_1$. This is also why we put $\delta_1 r_1^d$ (and not $\delta_1 r^d$) in the right-hand side.

Incidentally, the statement below would still be true if we replaced the ball $B(x, r)$ with a cube, or some other object that is bilipschitz-equivalent to a ball (and modified the annulus $A_{\eta r_1}(x, r)$ accordingly), but hopefully we shall not need to know this.

Theorem 3.8. *If the nonempty closed set $E \subset \mathbb{R}^n$ satisfies (1.2) (for some choice of $r_0 > 0$ and $\varepsilon_0 > 0$) and TND , then $\mathcal{H}\dim(E) \geq d^*$ for some $d^* > d$.*

As we shall see at the beginning of Section 7, we shall only need to apply TND with one value of C_1 , which depends only on n and d , and then d^* will depend only on n , d , ε_0 , and the values of α , η , and δ_1 that we get by applying TND for this choice of C_1 .

In the rest of this section, we want to show that Theorem 1.8 follows from Theorem 3.8. Hopefully, this will also convince the reader that TND is not too difficult to prove. The later sections will be devoted to the proof of Theorem 3.8.

So let E satisfy the hypotheses of Theorem 1.8, and suppose in addition that

$$(3.9) \quad \mathcal{H}\dim(E) < d + 1$$

(otherwise, there is nothing to prove). We want to show that $E \in TND$, so we give ourselves a large constant C_1 , and we try to find positive constants so that the condition in TND holds. For future reference, let us announce already that we shall keep the same value for r_0 and choose

$$(3.10) \quad \alpha = \frac{1}{2} \text{Min}(\eta_0, \alpha_0), \quad \eta = 2^{-d-1} 10^{-1} C_1^{-1} \delta_0, \quad \text{and} \quad \delta_1 = 2^{-d-1} \delta_0.$$

Let x_1 , r_1 , and ρ be as in (3.3), and assume that $H^d(E_\rho \cap B(x_1, r_1)) \leq C_1 r_1^d$, as in (3.4). We take $x = x_1$ and choose r such that

$$(3.11) \quad r_1/2 \leq r \leq 2r_1/3$$

and the mass of E_ρ in the thin annulus $A_{\eta r_1}(x, r) = B(x, r) \setminus B(x, r - \eta r_1)$ is very small. More precisely, note that we can find more than $(10\eta)^{-1}$ choices of r such that (3.11) holds and the $A_{\eta r_1}(x, r)$ are disjoint, so we can pick r such that

$$(3.12) \quad \begin{aligned} H^d(E_\rho \cap A_{\eta r_1}(x, r)) &\leq 10\eta H^d(E_\rho \cap B(x_1, r_1)) \\ &\leq 10\eta C_1 r_1^d \leq 2^{-d-1} \delta_0 r_1^d \leq \delta_1 r_1^d, \end{aligned}$$

by (3.4) and by our choice of η and δ_1 in (3.10).

So we picked $x \in E$ and r such that $B(x, r) \subset B(x_1, r_1)$. What we need to do now is take a family $\{\varphi_t\}$ of Lipschitz functions on \mathbb{R}^n that satisfies (1.3), (1.4), (1.5), and (3.5), and show that it satisfies (3.6). By (3.12), it will be enough to check that

$$(3.13) \quad H^d(\varphi_1(E_\rho \cap B(x, r))) \geq 2\delta_1 r_1^d.$$

To do so, we want to use $\{\varphi_t\}$ to construct a deformation $\{f_t\}$ of E , and then apply the hypotheses of Theorem 1.8 to $\{f_t\}$. So it is a good idea to first deform E into a subset of E_ρ .

Call $\mathcal{Q}_1(\rho)$ the set of cubes $Q \in \mathcal{Q}(\rho)$ such that Q meets $B(x, r)$, $\mathcal{Q}_2(\rho)$ the set of cubes $Q \in \mathcal{Q}(\rho)$ that meet a cube of $\mathcal{Q}_1(\rho)$, and set $D_j = \bigcup_{Q \in \mathcal{Q}_j(\rho)} Q$ for $j = 1, 2$.

Lemma 3.14. *There is a Lipschitz function $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$(3.15) \quad \psi(x) = x \quad \text{for } x \in \mathbb{R}^n \setminus D_2,$$

$$(3.16) \quad \psi(Q) \subset Q \quad \text{for } Q \in \mathcal{Q}(\rho),$$

and

$$(3.17) \quad \psi(E) \cap Q \subset S_d(Q) \quad \text{for } Q \in \mathcal{Q}_1(\rho).$$

The construction of ψ is a standard Federer-Flemming argument. We shall obtain ψ as a composition of various projection mappings onto skeletons of smaller and smaller dimensions. Let us give a rapid proof here; the reader may also look at Proposition 3.1 in [DS3], where a variant of the lemma is proved with more details.

We start with a “projection” π_1 that will send $E \cap D_2$ into faces of dimension $n - 1$. Let us first define π_1 on Q when Q is a cube of $\mathcal{Q}_2(\rho)$. Choose a point $c_Q \in \text{int}(Q) \setminus E$ (which is possible by (3.9)), and decide already that the restriction of π_1 to $[Q \cap E] \cup \partial Q$ will be the radial projection on ∂Q centered at c_Q . That is, $\pi_1(y)$ is the intersection of ∂Q with the half line through y that emanates from c_Q . Then extend π_1 to Q , so that $\pi_1(Q) \subset Q$ and π_1 is Lipschitz on Q . The Lipschitz constant may be enormous, if c_Q is very close to E , but this does not matter. Notice that we get a coherent definition of π_1 on D_2 . Indeed, if Q and $Q' \in \mathcal{Q}_2(\rho)$ meet, then both definitions yield $\pi_1(y) = y$ on $Q \cap Q'$, because $Q \cap Q'$ is contained in ∂Q and in $\partial Q'$. We take $\pi_1(y) = y$ out of D_2 . This is still coherent with our definition of π_1 on D_2 .

So we constructed a first Lipschitz function π_1 on \mathbb{R}^n such that $\pi_1(Q) \subset Q$ for $Q \in \mathcal{Q}(\rho)$ and for which $\pi_1(E)$ is contained in the union of $E \cap (\mathbb{R}^n \setminus D_2)$ and of the $(n - 1)$ -dimensional faces of the cubes Q , $Q \in \mathcal{Q}_2(\rho)$. Note also that $\mathcal{H} \dim(\pi_1(E)) < d + 1$, by (3.9) and because π_1 is Lipschitz.

If $d = n - 1$, we can take $\psi = \pi_1$. Otherwise, we have to continue the construction. We want to send points of the $(n - 1)$ -dimensional faces of the cubes Q , $Q \in \mathcal{Q}_1(\rho)$, to the boundary of these faces. Notice that when we talk about the boundary ∂F and the interior $\text{int}(F)$ of an

m -dimensional face F , we shall really see F as a m -dimensional object. Thus ∂F is a union of faces of dimension $m - 1$.

For each $(n - 1)$ -dimensional face F contained in D_1 , we choose a point $c_F \in \text{int}(F) \setminus \pi_1(E)$. Such a point exists because $\mathcal{H} \dim(\pi_1(E)) < d + 1 \leq n - 1$.

We define π_2 first on $\partial F \cup (\pi_1(E) \cap F)$, where we demand that $\pi_2(y)$ coincides with the radial projection on ∂F with center c_F . Then we extend this to a Lipschitz function defined on F , and such that $\pi_2(F) \subset F$. We do this on each $(n - 1)$ -face F contained in D_1 , and this gives a definition of π_2 on $\bigcup_{Q \in \mathcal{Q}_1(\rho)} S_{n-1}(Q)$. This definition is coherent, because π_2 is the identity on the intersections of $(n - 1)$ -faces of cubes $Q \in \mathcal{Q}_1(\rho)$.

Then we define π_2 on the $(n - 1)$ -faces F' of cubes $Q \in \mathcal{Q}_2(\rho)$ such that $F' \not\subset D_1$. We set $\pi_2(y) = y$ on F' ; notice that this is coherent with our definition on the faces $F \subset D_1$, because $F \cap F' \subset \partial F$ for such a combination of faces. This gives a definition of π_2 on all the faces of dimension $n - 1$ in D_2 . We extend this to the whole D_2 , making sure that $\pi_2(Q) \subset Q$ for $Q \in \mathcal{Q}_2(\rho)$ and $\pi_2(y) = y$ on ∂D_2 . Then we set $\pi_2(y) = y$ out of D_2 , and we get a Lipschitz map π_2 on \mathbb{R}^n .

Now $\pi_2 \circ \pi_1(E)$ is contained in $\mathbb{R}^n \setminus D_2$, plus the interior of the hyperfaces F of cubes $Q \in \mathcal{Q}_2(\rho)$ such that $F \not\subset D_1$ (i.e., that are not contained in a cube of $\mathcal{Q}_1(\rho)$), plus the faces of dimension $n - 2$ of cubes $Q \in \mathcal{Q}_1(\rho)$. This comes from our earlier description of $\pi_1(E)$, and the fact that we moved all the points of $\pi_1(E)$ that lied on a $(n - 1)$ -face $F \subset D_1$ to a boundary ∂F . Also, $\mathcal{H} \dim(\pi_2 \circ \pi_1(E)) \leq \mathcal{H} \dim(\pi_1(E)) < d + 1$, because π_2 is Lipschitz.

If $d = n - 2$, we can take $\psi = \pi_2 \circ \pi_1$; see the short justification below. Otherwise, we still need to project. We look first at faces of dimension $n - 2$ of cubes $Q \in \mathcal{Q}_1(\rho)$. For each such face F , we pick a point $c_F \in \text{int}(F) \setminus \pi_2 \circ \pi_1(E)$. Such a point exists, because $\mathcal{H} \dim(\pi_2 \circ \pi_1(E)) < d + 1 \leq n - 2$. We decide that on $\partial F \cup [\pi_2 \circ \pi_1(E)]$, π_3 coincides with the radial projection on ∂F with center c_F . On the other faces of dimension $n - 2$, we set $\pi_3(y) = y$ (which is coherent).

Then we extend π_3 to the faces of dimension $n - 1$. When we do this, we make sure that $\pi_3(F) \subset F$ for each such face, that $\pi_3(y) = y$ on the faces that do not touch D_1 , and also that the interior of F is sent to the interior of F . Then we extend π_3 to D_2 , still making sure that $\pi_3(Q) \subset Q$ for $Q \in \mathcal{Q}(\rho)$. Finally we set $\pi_3(y) = y$ out of D_2 .

Now $\pi_3 \circ \pi_2 \circ \pi_1(E)$ is contained in $\mathbb{R}^n \setminus D_2$, plus the interior of the hyperfaces F of cubes $Q \in \mathcal{Q}_2(\rho)$ such that $F \not\subset D_1$, plus the faces of dimension $n - 3$ of cubes $Q \in \mathcal{Q}_1(\rho)$. Indeed, if $z \in \pi_2 \circ \pi_1(E)$, then either

$z \in \mathbb{R}^n \setminus D_2$ and $\pi_3(z) = z$, or z lies in the interior of some hyperface F of a cube $Q \in \mathcal{Q}_2(\rho)$, with $F \not\subset D_1$, and then $\pi_3(z)$ also lies in the interior of F , or else z lies in a face of dimension $n-2$ of a cube $Q \in \mathcal{Q}_1(\rho)$, and then $\pi_3(z)$ lies on the boundary of that face.

We continue this procedure until we get to d -dimensional faces. Each time, we first construct π_k on the faces F of dimension $n-d+1$ of cubes $Q \in \mathcal{Q}_1(\rho)$. There we chose a center $c_F \in \text{int}(F) \setminus \pi_{k-1} \circ \dots \circ \pi_1(E)$, and decide that for $y \in \partial F \cup [\pi_{k-1} \circ \dots \circ \pi_1(E) \cap F]$, $\pi_k(y)$ is the radial projection on ∂F of y (with center c_F). We set $\pi_k(y) = y$ on the other faces of dimension $n-d+1$. Then we extend π_k to faces F' of larger and larger dimensions, making sure that $\pi_k(F') \subset F'$ for all F' , and that $\pi_3(y) = y$ on the faces that do not touch D_1 . When we get to hyperfaces, we also make sure that $\pi_k(\text{int}(F')) \subset \text{int}(F')$ for the hyperfaces F' that are contained in D_2 , but not in D_1 .

Eventually we get to $k = n-d$, and then $\pi_k \circ \dots \circ \pi_1(E)$ is contained in the union of the d -dimensional faces of cubes $Q \in \mathcal{Q}_1(\rho)$, plus the interior of some $n-1$ -dimensional faces that are not contained in a cube of $\mathcal{Q}_1(\rho)$, plus $\mathbb{R}^n \setminus D_2$. Thus (3.17) holds (for $\psi = \pi_k \circ \dots \circ \pi_1$), and the other properties (3.15) and (3.16) are true by construction. \square

Variants of Lemma 3.14 are often used, where we also require that for $Q \in \mathcal{Q}(\rho)$, $H^d(\psi(E \cap Q)) \leq CH^d(E \cap Q)$. Such a control is obtained by selecting more carefully the points c_F in the various faces, with a nice Fubini argument. But for the moment we do not need such refinements.

Now we use Lemma 3.14 to construct a deformation $\{\psi_t\}$ of E onto E_ρ . Set $\psi_t(y) = ty + (1-t)\psi(y)$ for $0 \leq t \leq 1$; this gives a one-parameter family of Lipschitz functions ψ_t , which all satisfy (3.15) and (3.16). In particular,

$$(3.18) \quad |\psi_t(y) - y| \leq \sqrt{n}\rho \text{ for } y \in \mathbb{R}^n \text{ and } 0 \leq t \leq 1.$$

We may now return to our be a one-parameter family of deformations $\{\varphi_t\}_{0 \leq t \leq 1}$, as in the definition of *TND*. Define new Lipschitz mappings f_t , $0 \leq t \leq 1$, on \mathbb{R}^n by

$$(3.19) \quad \begin{cases} f_t(y) = \psi_{2t}(y) & \text{for } 0 \leq t \leq 1/2, \\ f_t(y) = \varphi_{2t-1}(\psi(y)) & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

We want to check that $\{f_t\}$ satisfies the conditions (1.3)–(1.6), relative to the slightly larger ball $B = B(x, (1+\eta_0)r)$. First, each f_t is clearly

Lipschitz. Next, let us check that $f_t(\overline{B}) \subset \overline{B}$, as in (1.3). Observe that

$$(3.20) \quad B(x, r) \subset D_1 \subset D_2 \subset B(x, r + 2\sqrt{n}\rho) \subset B(x, r + \alpha r_1) \subset B$$

by definition of D_1 and D_2 , (3.3), (3.11), and because $\alpha \leq \frac{\eta_0}{2}$ by (3.10).

If $y \in D_2$, then $\psi_t(y) \in D_2$ for all t , by (3.16), and so (3.20) says that $f_s(y) \in B$ for $s \leq 1/2$. If in addition $\psi(y) \in \overline{B}(x, r)$, then $\varphi_t(\psi(y)) \in \overline{B}(x, r)$, by (1.3) for φ_t , and $f_s(y) = \varphi_{2s-1}(\psi(y)) \in \overline{B}(x, r) \subset B$ for $s \geq 1/2$, by (3.20). If instead $\psi(y) \notin \overline{B}(x, r)$, then (1.5) says that $\varphi_t(\psi(y)) = \psi(y)$, so $f_s(y) = \psi(y)$ for $s \geq 1/2$. Thus $f_s(y) \in D_2 \subset B$, as for $s = 1/2$ above. So $f_t(D_2) \subset B$, and we are left with $\overline{B} \setminus D_2$. But

$$(3.21) \quad \psi_t(y) = y \text{ and } f_t(y) = y \text{ for all } t \text{ when } y \in \mathbb{R}^n \setminus D_2,$$

by (3.15) (for ψ_t), and then (1.5) (because $\psi_t(y) = y$ lies out of $B(x, r)$, by (3.20)). Thus $f_t(\overline{B} \setminus D_2) \subset \overline{B}$ too, and f_t satisfies (1.3).

Note that (1.4) (the continuity of $f_t(y)$ in t) holds because $\psi_1 = \psi$ and $\varphi_0(y) = y$. Also, $f_0(y) = y$ because $\psi_0(y) = y$, and $f_t(y) = y$ for $y \notin B$, by (3.20) and (3.21). Which takes care of (1.5).

We still need to check (1.6). So let $y \in E \cap B$ and $0 \leq s \leq 1$ be given; we want to show that

$$(3.22) \quad \text{dist}(f_s(y), E) \leq \alpha_0(1 + \eta_0)r.$$

First,

$$(3.23) \quad \text{dist}(\psi_t(y), E) \leq |\psi(y) - y| \leq \sqrt{n}\rho \leq \frac{1}{2}\alpha r_1 \leq \alpha r \leq \frac{1}{2}\alpha_0 r$$

by (3.18), (3.3), (3.11), and (3.10). So (3.22) already holds for $0 \leq s \leq 1/2$.

For $s > 1/2$ we shall distinguish between cases again. Observe that

$$(3.24) \quad \psi(y) \notin B(x, r) \text{ when } y \in \mathbb{R}^n \setminus D_1,$$

because y lies in a cube $Q \in \mathcal{Q}(\rho) \setminus \mathcal{Q}_1(\rho)$, and then (3.16) says that $\psi(y)$ lies in Q , which does not meet $B(x, r)$ because $B(x, r) \subset \text{int}(D_1)$ (by (3.20) and because $B(x, r)$ is open).

If $y \in E \setminus D_1$ and $s \geq 1/2$, (3.24) says that $\psi(y) \notin B(x, r)$, then $f_s(y) = \varphi_{2s-1}(\psi(y)) = \psi(y)$, by (1.5) for $\{\varphi_t\}$. Then (3.22) follows from (3.23).

We are left with the case when $y \in D_1$ (and $s \geq 1/2$). Let us first check that

$$(3.25) \quad \psi(y) \in E_\rho \text{ when } y \in E \cap D_1.$$

Let Q be a cube of $\mathcal{Q}_1(\rho)$ that contains x . Then $\psi(y) \in S_d(Q)$ by (3.17). Since Q touches E (because $x \in E$), $S_d(Q) \subset E_\rho$ and (3.25) follows.

If in addition $\psi(y) \in B(x, r)$, we can apply (3.5) for $\{\varphi_t\}$ and get that $f_s(y) = \varphi_{2s-1}(\psi(y))$ lies within αr_1 of E . So (3.22) holds in that case, again because $\alpha r_1 \leq \alpha_0 r$. If $\psi(y) \notin B(x, r)$, then $f_s(y) = \psi(y)$ and (3.22) holds because of (3.23), just like when $s \leq 1/2$.

So $\{f_t\}$ satisfies (1.3)–(1.6). Since E satisfies the hypotheses of Theorem 1.8, the analogue of (1.7) for $B(x, (1 + \eta_0)r)$ holds (because $(1 + \eta_0)r \leq r_0$ by (3.3) and (3.11)). That is,

$$(3.26) \quad H^d[B(x, r) \cap f_1(E)] \geq H^d[B(x, (1 - \eta_0)(1 + \eta_0)r) \cap f_1(E)] \\ \geq \delta_0((1 + \eta_0)r)^d.$$

On the other hand, we have already seen that $f_1(y)$ lies out of $B(x, r)$ when $y \in \mathbb{R}^n \setminus D_1$ (by (3.24) and (1.5) for $\{\varphi_t\}$). So

$$(3.27) \quad B(x, r) \cap f_1(E) \subset f_1(E \cap D_1) = (\varphi_1 \circ \psi)(E \cap D_1) \subset \varphi_1(E_\rho),$$

where the last inclusion comes from (3.25). Hence

$$(3.28) \quad H^d(B(x, r) \cap \varphi_1(E_\rho)) \geq \delta_0 r^d \geq 2^{-d} r_1^d \delta_0 \geq 2\delta_1 r_1^d,$$

by (3.26), (3.11), and (3.10). And $H^d(\varphi_1(B(x, r) \cap E_\rho))$ is even larger, since $\varphi_1(y)$ lies out of $B(x, r)$ when $y \notin B(x, r)$.

This completes our proof of (3.13); as was said before (3.13), we get that E satisfies *TND* (as soon as it satisfies (3.9) and the topological condition in Theorem 1.8). Thus Theorem 1.8 follows from Theorem 3.8. \square

4. A functional on unions of faces

Let E be a closed set that satisfies the hypotheses of Theorem 3.8. We want to prove that its Hausdorff dimension is at least $d^* > d$, and the main point will be to establish the following lemma.

Lemma 4.1. *For every large $C_1 > 0$, we can find $\alpha_2 > 0$ and $r_0 > 0$ such that*

$$(4.2) \quad H^d(E_\rho \cap B(x_1, r_1)) > C_1 r_1^d$$

for all choices of $x_1 \in E$, $r_1 \in (0, r_0)$, and $\rho > 0$ such that $\rho \leq \alpha_2 r_1$.

We shall see in Section 7 that Theorem 3.8 is relatively easy and straightforward once we prove the lemma. And in fact it would be enough to prove the lemma for some (large) C_1 that can be computed from the dimension.

Our strategy for proving the lemma (which will keep us busy in Sections 4–6) will be to assume that (4.2) fails (so that (3.4) holds) and produce a contradiction. For this we shall use (3.4) and *TND* to construct an Ahlfors-regular, uniformly rectifiable set F^* with a big intersection

with E_ρ . The contradiction will then come from the fact that F^* is fairly flat at most scales and locations (by uniform rectifiability), while our assumption (1.2) says that E (and hence E_ρ , at least at scales larger than ρ) is never flat.

To find F^* , we shall minimize a functional like $J(F) = H^d(F \cap E_\rho) + MH^d(F \setminus E_\rho)$, where M is a large constant, and among sets F that are a finite union F^* of d -dimensional faces of cubes, plus a set of lower dimension $F \setminus F^*$, and that can be obtained from E_ρ by deformations like the $\{\varphi_t\}$ in the definition of TND . The point of taking finite unions of faces is that the existence of a minimizer F will be essentially trivial.

Our hypothesis TND will be used to find a ball $B(x, r) \subset B(x_1, r_1)$ where to define the functional, and then prove that $H^d(F)$ is never too small (by (3.6)). Then, if M is large enough (compared to δ_1^{-1}), we will get that $H^d(F^* \cap E_\rho) = H^d(F \cap E_\rho)$ is not too small either (because $H^d(F \setminus E_\rho)$ is very small).

To show that F^* is Ahlfors-regular and uniformly rectifiable, we shall prove that F is a quasiminimal set (with the terminology of Almgren, a restricted set) in some open set, and then use the main result in [DS3]. The quasiminimality constant, and then the Ahlfors-regularity and uniform rectifiability constants will depend on M , hence essentially all the constants at hand (except α_2), but we shall still be able to get a contradiction at some scale larger than ρ , if ρ/r_1 (or equivalently α_2) is small enough.

A good part of the argument below is quite similar to parts of [DS3, Section 12], but with small differences that seem to make it hard to import the results of [DS3] directly. So we shall repeat some of the arguments here, mainly for the convenience of the reader.

We start with the preparation of an appropriate domain and the definition of a functional. Since we are given $C_1 > 0$, we get other constants r_0 , α , η and δ_1 from TND . Let

$$(4.3) \quad x_1 \in E, \ 0 < r_1 < r_0, \text{ and } 0 < \rho \leq \alpha_2 r_1$$

be given, as in the statement of Lemma 4.1, and where α_2 will be chosen near the end of the argument. We suppose that (4.2) fails (or equivalently that (3.4) holds) and we want a contradiction.

By TND , we can find $x \in E$ and $r > 0$ such that $B(x, r) \subset B(x_1, r_1)$ and (3.6) holds for all the appropriate deformations $\{\varphi_t\}$.

We want to define a domain $V \subset B(x, r)$ where things will really happen. Since we shall work with lots of slightly different security balls

slightly smaller than $B(x, r)$, it will be convenient to set

$$(4.4) \quad B_j = B\left(x, r - \eta r_1 + \frac{j\eta r_1}{10}\right) \text{ for } 0 \leq j \leq 10.$$

Recall that $\eta r_1 \leq r$ (because (3.6) holds for the identity mapping, see the comment a little below (3.7)). So there is no difficulty with the definitions in (4.4). Also observe that $B_0 \subset \cdots \subset B_{10}$, and that the annulus from (3.7) is

$$(4.5) \quad A_{\eta r_1}(x, r) = B(x, r) \setminus B_0 = B_{10} \setminus B_0.$$

Things will be easier if the domain V is composed of (multiples) of dyadic cubes. First choose a number L such that

$$(4.6) \quad \frac{1}{200} \text{Min}(\eta, \alpha) r_1 \leq \sqrt{n} L \leq \frac{1}{100} \text{Min}(\eta, \alpha) r_1,$$

and such that $\rho^{-1}L$ is a power of 2. Call $\mathcal{Q}(L)$ the set of “dyadic cubes” of sidelength L , as in (3.1). Set

$$(4.7) \quad \mathcal{Q}_D = \{Q \in \mathcal{Q}(L); Q \cap B_6 \neq \emptyset\},$$

$$(4.8) \quad \mathcal{Q}_V = \left\{Q \in \mathcal{Q}_D; \text{dist}(Q, E) \leq \frac{\alpha r_1}{2}\right\},$$

$$(4.9) \quad D = \bigcup_{Q \in \mathcal{Q}_D} Q, \quad \text{and} \quad V = \bigcup_{Q \in \mathcal{Q}_V} Q.$$

Notice that $\text{diam}(Q) = \sqrt{n} L \leq 100^{-1} \text{Min}(\eta, \alpha) r_1$ for $Q \in \mathcal{Q}(L)$, by (4.6), so

$$(4.10) \quad B_6 \subset D \subset B_7 \subset B(x, r),$$

and also

$$(4.11) \quad \text{dist}(y, E) \leq \frac{\alpha r_1}{2} + \sqrt{n} L < \alpha r_1 \text{ for } y \in V.$$

Let us already decide that we shall take $\alpha_2 < (200\sqrt{n})^{-1} \text{Min}(\eta, \alpha)$, to make sure that in the following construction,

$$(4.12) \quad \rho \leq L$$

by (4.6). Note that each cube of $\mathcal{Q}(\rho)$ is neatly contained in a cube of $\mathcal{Q}(L)$, because $\rho^{-1}L$ is a power of 2. We shall work for a long time with a given ρ , but it will be important to make sure that our various constants do not depend on ρ (or α_2), because our final contradiction will require ρ to be small enough compared to r_1 .

Consider the class \mathcal{F}_0 of closed subsets of D that are a (finite) union of d -dimensional faces of cubes $Q \in \mathcal{Q}(\rho)$, plus perhaps a set of Hausdorff

dimension $\leq d - 1$. For $F \in \mathcal{F}_0$, denote by F^* the union of the faces of dimension d (of cubes in $\mathcal{Q}(\rho)$) that are contained in F . Thus

$$(4.13) \quad F^* \subset F \text{ and } \mathcal{H} \dim(F \setminus F^*) \leq d - 1.$$

We are mostly interested in the subclass \mathcal{F} of the sets $F \in \mathcal{F}_0$ that can be written

$$(4.14) \quad F = \varphi_1(E_\rho \cap D),$$

where $\{\varphi_t\}$, $0 \leq t \leq 1$, is a family of Lipschitz mapping on \mathbb{R}^n such that

$$(4.15) \quad \varphi_t(D) \subset D \text{ for all } t,$$

$$(4.16) \quad \varphi_t(y) \text{ is a continuous function of } t \text{ for every } y \in \mathbb{R}^n,$$

$$(4.17) \quad \varphi_t(y) = y \text{ for } t = 0 \text{ and for } y \in \mathbb{R}^n \setminus D,$$

$$(4.18) \quad \text{dist}(\varphi_t(y), E) < \alpha r_1 \text{ for } y \in E_\rho \cap D \text{ and } 0 \leq t \leq 1,$$

and even

$$(4.19) \quad \varphi_1(y) \in V \text{ for } y \in E_\rho \cap D.$$

Note that even though the set $F^* \setminus F$ has a lower Hausdorff dimension, it may play an important role in the description above. That is, (4.14) gives a representation of F as a deformation of $E_\rho \cap D$, but we do not necessarily have a similar representation of F^* , because for instance we cannot always deform F onto F^* .

Let us check that \mathcal{F} is not empty, and that

$$(4.20) \quad E_\rho \cap D \in \mathcal{F}.$$

The point is that we can take the trivial deformation $\varphi_t(y) = y$ in (4.15)–(4.19). Of course only (4.18) and (4.19) need to be checked. But

$$(4.21) \quad \text{dist}(y, E) \leq \sqrt{n} \rho \leq \sqrt{n} L \leq 100^{-1} \alpha r_1 \text{ for } y \in E_\rho,$$

by definition of E_ρ , (4.12) and (4.6). Hence

$$(4.22) \quad E_\rho \cap D = E_\rho \cap V,$$

by (4.8). So (4.18) and (4.19) hold, $\varphi_t(y) = y$ satisfies the required conditions, and $E_\rho \cap D \in \mathcal{F}$ (because it is a union of d -dimensional faces of cubes from $\mathcal{Q}(\rho)$).

Let us use *TND*. Each family $\{\varphi_t\}$ as in (4.15)–(4.19) satisfies (1.3), (1.4), and (1.5), because $D \subset B(x, r)$ (by (4.10)). Let us check that it also satisfies (3.5), i.e., that $\text{dist}(\varphi_t(y), E) \leq \alpha r_1$ for $y \in E_\rho \cap B(x, r)$ and $t \in [0, 1]$. If y lies out of D , then $\varphi_t(y) = y$, hence $\varphi_t(y) \in E_\rho$ and the result follows from (4.21). Otherwise, we can simply apply (4.18).

So (3.5) holds, we can apply *TND*, and (3.6) says that

$$(4.23) \quad H^d(\varphi_1(E_\rho \cap B(x, r))) \geq \delta_1 r_1^d + H^d(E_\rho \cap A_{\eta r_1}(x, r)).$$

Set $H = E_\rho \cap [B(x, r) \setminus D]$ for a few lines. Then $\varphi_1(E_\rho \cap B(x, r)) = \varphi_1(E_\rho \cap D) \cup \varphi_1(H)$ (because $D \subset B(x, r)$, see (4.10)), and this union is disjoint because $\varphi_1(D) \subset D$ and $\varphi_1(y) = y \notin D$ when y lies out of D . Moreover, $\varphi_1(H) = H$ (because $\varphi_1(y) = y$ on H), so the left-hand side of (4.23) is $H^d(\varphi_1(E_\rho \cap D)) + H^d(H)$. Also,

$$(4.24) \quad H = E_\rho \cap [B(x, r) \setminus D] \subset E_\rho \cap [B(x, r) \setminus B_6] \subset E_\rho \cap A_{\eta r_1}(x, r)$$

by (4.10) and (4.5) (or the definition (3.7)). So we can subtract $H^d(H)$ from both sides of (4.23), and we get that

$$(4.25) \quad H^d(\varphi_1(E_\rho \cap D)) \geq \delta_1 r_1^d + H^d(E_\rho \cap A_{\eta r_1}(x, r) \cap D).$$

This holds for every family $\{\varphi_t\}$ such that (4.15)–(4.19) hold. When we restrict to families where $F = \varphi_1(E_\rho \cap D) \in \mathcal{F}_0$ as in (4.14), we get that

$$(4.26) \quad H^d(F) = H^d(\varphi_1(E_\rho \cap D)) \\ \geq \delta_1 r_1^d + H^d(E_\rho \cap A_{\eta r_1}(x, r) \cap D) \text{ for } F \in \mathcal{F}.$$

Set $M = \frac{2C_1}{\delta_1}$, with C_1 as in (4.2) and (3.4), and define a functional on \mathcal{F} by

$$(4.27) \quad J(F) = H^d(F \cap E_\rho) + MH^d(F \setminus E_\rho) \text{ for } F \in \mathcal{F}.$$

Note that $J(F) = H^d(F^* \cap E_\rho) + MH^d(F^* \setminus E_\rho)$, where F^* still denotes the finite union of d -faces that satisfies (4.13). Since $\mathcal{F} \neq \emptyset$ (by (4.20)) and there is only a finite number of possible sets F^* , we can find a minimizer F , i.e., a set $F \in \mathcal{F}$ such that

$$(4.28) \quad J(F) = \inf \{J(F'); F' \in \mathcal{F}\}.$$

Recall from (4.20) that $F' = E_\rho \cap D \in \mathcal{F}$. Hence

$$(4.29) \quad J(F) \leq J(E_\rho \cap D) = H^d(E_\rho \cap D) \leq H^d(E_\rho \cap B(x, r)) \leq C_1 r_1^d,$$

by (4.10) and because we assumed that (4.2) does not hold (to get a contradiction). Then

$$(4.30) \quad H^d(F \setminus E_\rho) \leq M^{-1}J(F) \leq \frac{\delta_1}{2C_1}C_1 r_1^d = \frac{\delta_1 r_1^d}{2}$$

by (4.27), (4.29), and our definition of M . Now (4.26) yields

$$(4.31) \quad H^d(F \cap E_\rho) \geq H^d(F) - H^d(F \setminus E_\rho) \\ \geq \delta_1 r_1^d/2 + H^d(E_\rho \cap A_{\eta r_1}(x, r) \cap D).$$

Thus F has a fairly big intersection with E_ρ . It will be useful to have a lower bound on the size of the part of $F \cap E_\rho$ that lies in $B_0 = B(x, r - \eta r_1)$ (recall the notation (4.4)), because it will be easier to show that F^* is regular there. Note that $F \cap E_\rho$ is the disjoint union of $F \cap E_\rho \cap B_0$ and $F \cap E_\rho \setminus B_0$. Recall that $F \subset D \subset B(x, r)$ by definition of \mathcal{F}_0 and (4.10), so $F \cap E_\rho \setminus B_0 \subset E_\rho \cap A_{\eta r_1}(x, r) \cap D$, by (4.5). So we may subtract $H^d(F \cap E_\rho \setminus B_0)$ from both sides of (4.31) and we get that

$$(4.32) \quad H^d(F \cap E_\rho \cap B_0) \geq \delta_1 r_1^d / 2.$$

Our next goal is to show that F^* is locally uniformly rectifiable in B_0 because F is a minimizer for some nice functional. This will be used to produce lots of balls where F (and hence E_ρ) are very flat, and a contradiction with (1.2) will ensue.

5. Almgren quasiminimality of F

Let F be the minimizer for J introduced in the last section, and set

$$(5.1) \quad B_2 = B\left(x, r - \frac{8\eta r_1}{10}\right)$$

(as in (4.4)). The goal of this section is to show that

$$(5.2) \quad S = F \cap B_2 \text{ is a } (B_2, k, \delta)\text{-quasiminimizer for } H^d,$$

where k is a constant that depends on C_1 and δ_1 (in fact, $k = CM = 2CC_1\delta_1^{-1}$ for some geometric constant C),

$$(5.3) \quad \delta = 4^{-n-1}L,$$

and the notion of quasiminimizer for H^d is the same as in [DS3] and [A1], and will be explained soon. See [DS3, pp. 7–8], for the official definition that we shall use.

What does (5.2) mean? First, S should be closed in B_2 and non empty (Condition (1.1) in [DS3]). This comes from (4.14) and (4.32) respectively. Next, $H^d(S)$ should be locally finite (Condition (1.2) in [DS3]), which is obvious because $H^d(F) \leq J(F) < +\infty$. And then S should satisfy (1.8) in [DS3].

For this, we give ourselves a Lipschitz function $\psi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and that satisfies the conditions (1.5)–(1.7) in [DS3]. In particular, (1.5) and (1.6) says that if we set

$$(5.4) \quad W = \{z \in \mathbb{R}^n; \psi(z) \neq z\},$$

then

$$(5.5) \quad \text{diam}(W \cup \psi(W)) < \delta$$

and

$$(5.6) \quad W \cup \psi(W) \in B_2,$$

i.e., $\text{dist}(W \cup \psi(W), \mathbb{R}^n \setminus B_2) > 0$. The condition (1.7) in [DS3] says that there is a homotopy from the identity to f , with conditions like (5.5) and (5.6), but we shall not need it. We need to show that under these conditions,

$$(5.7) \quad H^d(S \cap W) \leq kH^d(\psi(S \cap W)).$$

Our strategy for proving this is to use ψ to construct a competitor $F_1 \in \mathcal{F}$, and then deduce (5.7) from the minimality of $J(F)$. Of course we cannot take $F_1 = \psi(F)$ directly, because it probably does not lie in \mathcal{F} , so we shall have to modify it in two ways.

First, it could be that $\psi(F)$ gets out of V , which would be bad because of our constraint (4.19) on deformations. To fix this problem, we shall construct a Lipschitz retraction onto V , and use it to push back $\psi(F)$ inside V .

But also, $\psi(F)$ is not necessarily a finite union of d -faces, plus a set of lower dimension, so we shall need to project it onto d -dimensional skeletons (a little like in Lemma 3.14).

We start with the Lipschitz retraction.

Lemma 5.8. *If V is a finite union of (closed) dyadic cubes of side-length L , there is 4^n -Lipschitz function h defined on*

$$(5.9) \quad V_+ = \left\{ y \in \mathbb{R}^n; \text{dist}(y, V) \leq \frac{L}{4} \right\},$$

and such that

$$(5.10) \quad h(V_+) \subset V,$$

$$(5.11) \quad h(y) = y \quad \text{for } y \in V,$$

and

$$(5.12) \quad |h(y) - y| \leq \sqrt{n}L \quad \text{for } y \in V_+.$$

The statement is not exactly the same as in Lemma 12.2 in [DS3], but the proof will be the same. We shall get h as a composition of a few mappings g_j .

Set $g_1(y) = y$ on V . Call \mathcal{Q}_+ the collection of cubes $Q \in \mathcal{Q}(L)$ that touch V but are not contained in it. Thus

$$(5.13) \quad V_+ \setminus V \subset \bigcup_{Q \in \mathcal{Q}_+} Q \cap V_+.$$

Call c_Q the center of $Q \in \mathcal{Q}_+$. Then $\text{dist}(c_Q, V_+) \geq \text{dist}(c_Q, V) - L/4 \geq L/4$. We define g_1 on $Q \cap V_+$ to be the radial projection on ∂Q with center c_Q (as defined in the proof of Lemma 3.14). Note that $g_1(y) = y$ on ∂Q , so our different definitions coincide (on intersections of cubes). Observe also that g_1 is 4-Lipschitz on V_+ , because each c_Q is far from V_+ . Finally, we claim that

$$(5.14) \quad \text{dist}(g_1(y), V) \leq \text{dist}(y, V) \leq 4^{-1}L \text{ for } y \in V_+.$$

This is trivial when $y \in V$. Then let $y \in V_+ \setminus V$ be given, let $Q \in \mathcal{Q}_+$ contain y , and let z be a point of V such that $|z - y| = \text{dist}(y, V)$. Also call Q_z a cube of V that contains z . Then Q_z touches Q (because otherwise it would be too far), and even $z \in Q \cap Q_z$ (by inspection). Note that c_Q lies further from $T = Q \cap Q_z$ than y . Then $\text{dist}(g_1(y), Q) \leq \text{dist}(g_1(y), T) \leq \text{dist}(y, T) \leq |z - y| = \text{dist}(y, V)$, as needed for (5.14). Now (5.14) yields

$$(5.15) \quad g_1(V_+) \subset V \cup \bigcup_{Q \in \mathcal{Q}_+} [S_{n-1}(Q) \cap V_+].$$

We set $g_2(y) = y$ on V . For each $(d-1)$ -face T of a cube $Q \in \mathcal{Q}_+$ such that T is not already contained in V , we decide that for $y \in T \cap V_+$, $g_2(y)$ is the radial projection of y on ∂T (centered at the center of T). This definition is coherent, because T only meets different faces of V along ∂T , where g_2 is the identity. Note that we do not need to extend g_2 , because it is already defined on $g_1(V_+)$, by (5.15). Also, g_2 is 4-Lipschitz, again because $y \in V_+$ is never close to the center of T .

As before, $\text{dist}(g_2(y), V) \leq \text{dist}(y, V) \leq 4^{-1}L$ for $y \in T \cap V_+$, and hence $g_2(y) \in S_{n-2}(T) \cap V_+$. Thus

$$(5.16) \quad g_2 \circ g_1(V_+) \subset V \cup \bigcup_{Q \in \mathcal{Q}_+} [S_{n-2}(Q) \cap V_+].$$

We can define g_3 on V and on the sets $T \cap V_+$, where T is a $(n-2)$ -dimensional face of some cube $Q \in \mathcal{Q}_+$ and is not contained in V , as we did before.

After a few iterations, we get a 4^n -Lipschitz mapping $h = g_1 \circ \dots \circ g_n$ that sends V_+ to V , plus the intersection of V_+ with a set of vertices. But only the vertices of V can lie in V^+ , so $h(V_+) \subset V$, as in (5.10). We have that $h(y) = y$ for $y \in V$ because all the g_j fix points of V , and our last condition (5.12) holds because $g_j(Q) \subset Q$ for each $Q \in \mathcal{Q}(L)$ and each j , so that if we start from y in some cube $Q \in \mathcal{Q}(L)$, $h(y)$ lies in the same Q . \square

We may now return to the Lipschitz mapping ψ of the beginning of this section. Recall from (4.14) and (4.19) that

$$(5.17) \quad F = \varphi_1(E_\rho \cap D) \subset V,$$

for some family $\{\varphi_t\}$ that satisfies (4.15)–(4.19). Notice that if $\psi(y) \neq y$, then $y \in W$ (by (5.4)), and then $|\psi(y) - y| \leq \text{diam}(W \cup \psi(W)) < \delta$ by (5.5). Hence

$$(5.18) \quad |\psi(y) - y| \leq \delta = 4^{-n-1}L \text{ for } y \in \mathbb{R}^n,$$

by (5.3). By (5.17) and (5.18), $\psi(y) \in V_+$ (and hence $h(\psi(y))$ is defined) for $y \in F$. Set

$$(5.19) \quad F_1 = \psi(F) \text{ and } F_2 = h(F_1).$$

Thus $F_2 \subset V$. We still need to project it back on d -dimensional skeletons. For this part of the construction, we shall work on cubes of sidelength ρ . Some additional notation will be useful. Call $\mathcal{S}_d \subset \mathbb{R}^n$ the union of all the d -dimensional faces of cubes $Q \in \mathcal{Q}(\rho)$. With the notations of the beginning of Section 3,

$$(5.20) \quad \mathcal{S}_d = \bigcup_{Q \in \mathcal{Q}(\rho)} \mathcal{S}_d(Q).$$

For $A \subset \mathbb{R}^n$, call $\mathcal{N}(A)$ the union of all the cubes $Q \in \mathcal{Q}(\rho)$ which meet a cube of $\mathcal{Q}(\rho)$ which meets A .

Let W be as in (5.4), and set

$$(5.21) \quad X = h(\psi(F \cap W)).$$

This is the part that we may need to project back onto \mathcal{S}_d . Indeed

$$(5.22) \quad h(\psi(y)) = \psi(y) = y \text{ for } y \in F \setminus W,$$

by (5.4), (5.17), and (5.11), so $h(\psi(F \setminus W)) = F \setminus W$. Except for a set of lower dimension, this set is contained in F^* , which itself is contained in \mathcal{S}_d by definition of \mathcal{F}_0 .

We may now apply Lemma 11.14 on [DS3, p. 90] to X . The notations are almost the same, except that ρ is called 2^{-j} there, and what we call $\mathcal{N}(A)$ here is denoted by $\mathcal{N}_j(A)$ there. We get the existence of a Lipschitz

function f on \mathbb{R}^n such that

$$(5.23) \quad f(z) = z \text{ out of } \mathcal{N}(\overline{X}),$$

$$(5.24) \quad f(z) = z \text{ for } z \in \mathcal{S}_d,$$

$$(5.25) \quad f(\overline{X}) \subset \mathcal{S}_d,$$

$$(5.26) \quad f(Q) \subset Q \text{ for } Q \in \mathcal{Q}(\rho),$$

and

$$(5.27) \quad H^d(f(\overline{X} \cap Q \setminus \mathcal{S}_d)) \leq CH^d(\overline{X} \cap Q \setminus \mathcal{S}_d) \text{ for every } Q \in \mathcal{Q}(\rho).$$

The proof of Lemma 11.14 in [DS3] is similar to the proof of Lemma 3.14 above, except that we also need (5.27) now. As was alluded to above, this can be obtained by choosing the centers c_F in the various faces more carefully, so that their average distance to the points of the current image of \overline{X} is not too small. \square

Set $F_3 = f(F_2) = f(h(\psi(F)))$, and let us already check that

$$(5.28) \quad \mathcal{H} \dim(F_3 \setminus \mathcal{S}_d) \leq d - 1.$$

We already know from (5.22) that $h(\psi(F^* \setminus W)) = F^* \setminus W \subset \mathcal{S}_d$, because $F^* \subset \mathcal{S}_d$ by definition of \mathcal{F}_0 . Then $f(h(\psi(F^* \setminus W))) \subset \mathcal{S}_d$ too, by (5.24). Since $f(h(\psi(F^* \cap W))) \subset f(X) \subset \mathcal{S}_d$ by definition (5.21) of X and by (5.25), we are left with $f(h(\psi(F \setminus F^*)))$, which has Hausdorff dimension at most $d - 1$ by (4.13) and because $f \circ h \circ \psi$ is Lipschitz.

Now (5.28) does not exactly say that $F_3 \in \mathcal{F}_0$, because $F_3 \cap \mathcal{S}_d$ is not necessarily an exact union of full d -faces. To fix this, we need to compose with yet another function π .

Call \mathcal{A} the set of d -dimensional faces of cubes in $\mathcal{Q}(\rho)$ that meet F_3 on a set of dimension $> d - 1$ but are not contained in F_3 . For $T \in \mathcal{A}$, choose an origin $c_T \in \text{int}(T) \setminus F_3$. Such a point exists by definition of \mathcal{A} , and $\text{dist}(c_T, F_3) > 0$ because F_3 is closed. For $z \in F_3 \cap T$, we let $\pi(z)$ be the radial projection on ∂T with center c_T . Then we extend this definition to T , so that π is Lipschitz on T , $\pi(T) \subset T$, and $\pi(z) = z$ on ∂T . We do this for each $T \in \mathcal{A}$, then set $\pi(z) = z$ on the rest of \mathcal{S}_d , and then extend π to \mathbb{R}^n so that it stays Lipschitz,

$$(5.29) \quad \pi(Q) \subset Q \text{ for every } Q \in \mathcal{Q}(\rho),$$

and that

$$(5.30) \quad \pi(z) = z \text{ on the cubes that do not touch a face } T \in \mathcal{A}.$$

The details are the same as in the previous similar constructions.

Our next middle-term goal is to show that

$$(5.31) \quad F_4 = \pi(F_3) = \pi \circ f \circ h \circ \psi(F)$$

lies in the set \mathcal{F} of competitors for J .

By definition of π , $\pi(F_3)$ is now an exact union of d -faces (precisely, the faces of \mathcal{S}_d that were contained in F_3), plus perhaps a set of dimension $\leq d-1$ coming from the almost-empty faces, and the boundaries of faces $T \in \mathcal{A}$ where we used π to project. And the remaining piece $\pi(F_3 \setminus \mathcal{S}_d)$ is still at most $(d-1)$ -dimensional, by (5.28). So we shall get that $F_4 \in \mathcal{F}_0$ as soon as we check that $F_4 \subset D$.

Recall that $F_2 = h(F_1) \subset V$, by (5.19) and (5.10). Since the mappings f and π preserve cubes of $\mathcal{Q}(\rho)$ (by (5.26) and (5.29)), $F_4 = \pi(f(F_2)) \subset V \subset D$ (by (4.8) and (4.9)), and

$$(5.32) \quad F_4 \in \mathcal{F}_0.$$

The verification that $F_4 \in \mathcal{F}$ will take some time, because we need to construct a family of Lipschitz functions as in (4.14)–(4.19). The following function g , which we first define on V , will be useful. Set

$$(5.33) \quad g(z) = \pi \circ f \circ h \circ \psi(z) \text{ for } z \in V.$$

The definition makes sense, because if $z \in V$, then $\text{dist}(\psi(z), V) < 4^{-n}L$ by (5.18), and then $h(\psi(z))$ is defined and lies in V , by (5.9) and (5.10). Then $g(z)$ is well defined, because f and π are defined everywhere. Note that in addition

$$(5.34) \quad g(z) \in V \subset D \text{ for } z \in V,$$

because f and π preserve cubes of $\mathcal{Q}(\rho)$, and $V \subset D$ by definition (see (4.8) and (4.9)).

We want to see how much g moves points. First observe that

$$(5.35) \quad h(\psi(z)) = \psi(z) = z \text{ for } z \in V \setminus B_2,$$

by (5.4), (5.6), and (5.11). Even when $z \in B_2$ we can still say something, since

$$(5.36) \quad |h(\psi(z)) - z| \leq |h(\psi(z)) - \psi(z)| + |\psi(z) - z| \\ \leq \sqrt{n}L + 4^{-n-1}L \leq 2\sqrt{n}L \text{ for } z \in V,$$

by (5.12) and (5.18).

Next we look at f and the set $\mathcal{N}(\overline{X})$ where it may move points. Recall from (5.21) that $X = h(\psi(F \cap W))$, which is defined because $F \subset V$ (by (5.17)). Also recall from (5.4) that $\psi(W) \subset B_2$, so

$$(5.37) \quad \text{dist}(h(\psi(z)), B_2) \leq |h(\psi(z)) - \psi(z)| \leq \sqrt{n}L \text{ for } z \in F \cap W,$$

by (5.12) again. In other words, $\text{dist}(y, B_2) \leq \sqrt{n}L$ for $y \in X$.

By definition, every point of $\mathcal{N}(\overline{X})$ lies within $2\sqrt{n}\rho$ of X , hence within $3\sqrt{n}L$ of B_2 (by (4.12)). Since $3\sqrt{n}L < \eta r_1/30$ by (4.6), we get that

$$(5.38) \quad \mathcal{N}(\overline{X}) \subset B_3.$$

Recall from (5.23) that $f(z) = z$ out of $\mathcal{N}(\overline{X})$. Then

$$(5.39) \quad f(h(\psi(z))) = h(\psi(z)) = \psi(z) = z \text{ for } z \in V \setminus B_3,$$

by (5.35), and (5.38).

Now we consider π . Recall from (5.30) that $\pi(z) = z$ unless $z \in Q$ for a cube $Q \in \mathcal{Q}(\rho)$ that touches a face $T \in \mathcal{A}$. Let us check that

$$(5.40) \quad \text{dist}(T, B_3) \leq \frac{\eta r_1}{30} \text{ for } T \in \mathcal{A}.$$

Let $T \in \mathcal{A}$ be given, and first assume that $T \subset F$. Since T is not contained in $F_3 = f(h(\psi(F)))$, it must contain some point $z \in F$ such that $f(h(\psi(z))) \neq z$. Since $z \in F \subset V$, (5.39) says that $z \in B_3$. In this case (5.40) holds trivially.

If A is not contained in F , $\mathcal{H} \dim(A \cap F) \leq d-1$ (because $F \in \mathcal{F}_0$), hence there are some points of $F_3 \cap T$ that do not lie in F . Such points are of the form $y = f(h(\psi(z)))$, with $z \in F$. Since $y \notin F$, $f(h(\psi(z))) \neq z$ and (5.39) says that $z \in B_3$. Also,

$$(5.41) \quad \begin{aligned} |f(h(\psi(z))) - z| &\leq |f(h(\psi(z))) - h(\psi(z))| + |h(\psi(z)) - z| \\ &\leq \sqrt{n}\rho + 2\sqrt{n}L \leq 3\sqrt{n}L \leq \frac{\eta r_1}{30} \end{aligned} \quad \text{for } z \in V,$$

by (5.26), (5.36), (4.12), and (4.6). When we apply this with $z \in F$ and $y = f(h(\psi(z))) \in F_3 \cap T$ as above, we get that $\text{dist}(T, B_3) \leq |z - y| = |z - f(h(\psi(z)))| \leq \eta r_1/30$, and (5.40) holds.

If $z \in \mathbb{R}^n$ is such that $\pi(z) \neq z$, (5.30) says that $z \in Q$ for a cube $Q \in \mathcal{Q}(\rho)$ that touches a face $T \in \mathcal{A}$; then $\text{dist}(z, B_3) \leq \eta r_1/30 + 2\sqrt{n}\rho < \eta r_1/10$ by (5.40), (4.12), and (4.6). Thus

$$(5.42) \quad \pi(z) = z \text{ for } z \in \mathbb{R}^n \setminus B_4,$$

and hence (5.39) and (5.33) yield

$$(5.43) \quad g(z) = f(h(\psi(z))) = h(\psi(z)) = \psi(z) = z \text{ for } z \in V \setminus B_4.$$

We are now ready to complete our definition of g .

Lemma 5.44. *We can find a Lipschitz function $g: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that (5.33) holds,*

$$(5.45) \quad g(z) = z \quad \text{for } z \in \mathbb{R}^n \setminus B_5,$$

and

$$(5.46) \quad g(z) \in \overline{B}_5 \subset D \quad \text{for } z \in \overline{B}_5.$$

Set $H = [V \cap B_5] \cup [\partial B_5]$. If we keep (5.33) on $V \cap B_5$ and set $g(z) = z$ on ∂B_5 , we have a coherent definition of g on H , by (5.43). Moreover, g is Lipschitz on H . Indeed, since g is clearly Lipschitz on V (by (5.33)) and on $\overline{B}_5 \setminus B_4$ (because it is the identity there), it is enough to estimate $g(y) - g(z)$ when $y \in V \cap B_4$ and $z \in \partial B_5$. But then $|g(y) - g(z)| \leq 2r_1$ because $g(y) \in V \subset B(x, r)$ (by (5.34) and (4.4)), while $|y - z| \geq \eta r_1/10$ (by (4.4) again). So g is Lipschitz on H .

Let us also check that $g(H) \subset \overline{B}_5$. When $z \in [V \setminus B_4] \cup \partial B_5$, $g(z) = z$ (by (5.43)), and so $g(z) \in \overline{B}_5$. Otherwise, notice that for $z \in V$,

$$(5.47) \quad \begin{aligned} |g(z) - z| &\leq |g(z) - f(h(\psi(z)))| + |f(h(\psi(z))) - z| \\ &\leq |\pi(f(h(\psi(z)))) - f(h(\psi(z)))| + 3\sqrt{n}L \\ &\leq \sqrt{n}\rho + 3\sqrt{n}L < 4\sqrt{n}L \leq \frac{\eta r_1}{20} \end{aligned}$$

by (5.33), (5.41), (5.29), (4.12), and (4.6). In the remaining case when $z \in V \cap B_4$, we still get that $g(z) \in \overline{B}_5$.

It is now easy to extend g from H to \overline{B}_5 , so that it stays Lipschitz and with values in \overline{B}_5 . [Otherwise, compose with a Lipschitz retraction on \overline{B}_5 .] Finally extend g to the rest of \mathbb{R}^n , by setting $g(z) = z$ out of \overline{B}_5 . We still get a Lipschitz function, and the properties (5.33), (5.45), and (5.46) hold by construction. Lemma 5.44 follows. \square

Recall that we want to use g to construct a family of Lipschitz functions as in (4.14)–(4.19) and show that $F_4 \in \mathcal{F}$. Denote by $\{\varphi_t\}$ the family associated to $F \in \mathcal{F}$ and set

$$(5.48) \quad \tilde{\varphi}_t(y) = \varphi_{2t}(y) \quad \text{for } 0 \leq t \leq 1/2$$

(and all $y \in \mathbb{R}^n$) and

$$(5.49) \quad \tilde{\varphi}_t(y) = (2 - 2t)\varphi_1(y) + (2t - 1)g(\varphi_1(y)) \quad \text{for } 1/2 < t \leq 1.$$

Each $\tilde{\varphi}_t$ is Lipschitz, because g and the φ_t are. Next we need to check that $\tilde{\varphi}_t(D) \subset D$, as in (4.15). When $t \leq 1/2$, this comes directly from (4.15) (for the φ_t). So we may restrict to $t > 1/2$.

Let $y \in D$ be given. If $\varphi_1(y) \in \mathbb{R}^n \setminus B_5$, (5.45) says that $g(\varphi_1(y)) = \varphi_1(y)$ and hence $\tilde{\varphi}_t(y) = \varphi_1(y) \in D$ (by (5.49) and (4.15) for φ_1). Otherwise $\varphi_1(y) \in B_5$, (5.46) says that $g(\varphi_1(y)) \in \overline{B}_5$, and $\tilde{\varphi}_t(y) \in \overline{B}_5 \subset D$ by (5.49), because \overline{B}_5 is convex, and by (4.10). So $\tilde{\varphi}_t(D) \subset D$ in all cases.

The continuity in t of $\tilde{\varphi}_t(y)$ (as in (4.16)) is clear. For the analogue of (4.17), we need to check that $\tilde{\varphi}_t(y) = y$ when $t = 0$ and when $y \in \mathbb{R}^d \setminus D$. When $t = 0$, this follows directly from (4.17). When $t \leq 1/2$ and $y \in \mathbb{R}^d \setminus D$, this is just (4.17) again. When $t > 1/2$ and $y \in \mathbb{R}^d \setminus D$, we observe that $\varphi_1(y) = y$ by (4.17), then $g(\varphi_1(y)) = y$ by (5.45) and (4.10), so $\tilde{\varphi}_t(y) = y$ by (5.49).

For (4.18), we need to check that $\text{dist}(\tilde{\varphi}_t(y), E) < \alpha r_1$ when $y \in E_\rho \cap D$ and $0 \leq t \leq 1$. When $t \leq 1/2$, this comes from (4.18) for the φ_t . For $t > 1/2$, first observe that

$$(5.50) \quad \varphi_1(y) \in V$$

by (4.19), so $\text{dist}(\varphi_1(t), E) \leq \frac{\alpha r_1}{2} + \sqrt{n}L$, by (4.11). Also, (5.47) says that $|g(\varphi_1(y)) - \varphi_1(y)| \leq 4\sqrt{n}L$. Since by (5.49) $\tilde{\varphi}_t(y)$ is a convex combination of $\varphi_1(y)$ and $g(\varphi_1(y))$, we get that $\text{dist}(\tilde{\varphi}_t(t), E) \leq \frac{\alpha r_1}{2} + 5\sqrt{n}L < \alpha r_1$, by (4.6) and as needed for the analogue of (4.18).

We also need to check (4.19), i.e., that $\tilde{\varphi}_1(y) \in V$ when $y \in E_\rho \cap D$. This follows from (5.50) and (5.34). So our family $\{\tilde{\varphi}_t\}$ satisfies (4.14)–(4.19). In addition, $F_4 = g(F) = g(\varphi_1(E_\rho \cap D)) = \tilde{\varphi}_1(E_\rho \cap D)$ by (5.31), (5.33), (4.14), and (5.49). Since we already know from (5.32) that $F_4 \in \mathcal{F}_0$, we get that $F_4 \in \mathcal{F}$ (at last!).

Recall that F is a minimizer for J . Then $J(F) \leq J(F_4)$. Also recall that $F_4 = \pi(F_3)$, where π was designed to leave untouched all the d -dimensional faces of \mathcal{S}_d , except the faces $T \in \mathcal{A}$ where $F_3 \cap T$ was sent to ∂T . So F_4 is contained in F_3 , except perhaps for the image by π of the $(d-1)$ -dimensional part of F_3 that did not live on \mathcal{S}_d (see (5.28)), and a few $(d-1)$ -dimensional faces ∂T . Thus $J(F_4) \leq J(F_3)$, and altogether

$$(5.51) \quad J(F) \leq J(F_3).$$

We want to analyze this carefully, so we cut F into the disjoint pieces $F \cap W$ and $F \setminus W$, where W is as in (5.4). Then

$$(5.52) \quad J(F) = J(F \cap W) + J(F \setminus W),$$

where for convenience we still set

$$(5.53) \quad J(A) = H^d(A \cap E_\rho) + MH^d(A \setminus E_\rho)$$

as in (4.27), even when $A \notin \mathcal{F}$.

For the next few lines, it will be convenient to set

$$(5.54) \quad \Phi = f \circ h \circ \psi.$$

Recall that $F_3 = \Phi(F)$, so $F_3 = \Phi(F \cap W) \cup \Phi(F \setminus W)$. For $y \in F \setminus W$, $h(\psi(y)) = y$ by (5.22). If in addition $y \in F^*$ (the finite union of faces that almost cover F , see (4.13)), then $y \in \mathcal{S}_d$, and (5.24) says that $f(y) = y$. Thus $\Phi(y) = y$ for $y \in F^* \setminus W$, and $\Phi(F^* \setminus W) = F^* \setminus W$. Since the small set $\Phi(F \setminus F^*)$ does not contribute, $J(F_3) \leq J(\Phi(F \cap W)) + J(F \setminus W)$. So (5.51) and (5.52) yield

$$(5.55) \quad J(F \cap W) \leq J(\Phi(F \cap W))$$

and then

$$(5.56) \quad H^d(F \cap W) \leq MH^d(\Phi(F \cap W)),$$

because of the formula (5.53) for J .

In view of (5.7), we would prefer to have $\psi(F \cap W)$ instead of $\Phi(F \cap W)$, so we want to show that h and f do not increase the measure of $\psi(F \cap W)$ too much. First, h is 4^n -Lipschitz (see Lemma 5.8), so

$$(5.57) \quad H^d(h \circ \psi(F \cap W)) \leq 4^{nd} H^d(\psi(F \cap W)).$$

Recall from (5.21) that $X = h \circ \psi(F \cap W)$, so the left-hand side of (5.57) is $H^d(X)$. We cut X into $X_1 = X \cap \mathcal{S}_d$ and $X_2 = X \setminus \mathcal{S}_d$. By (5.24), $f(z) = z$ on X_1 , so $H^d(f(X_1)) = H^d(X_1)$.

$$(5.58) \quad \begin{aligned} H^d(f(X_2)) &\leq \sum_{Q \in \mathcal{Q}(\rho)} H^d(f(X_2 \cap Q)) \\ &\leq C \sum_{Q \in \mathcal{Q}(\rho)} H^d(f(\overline{X} \cap Q \setminus \mathcal{S}_d)) \\ &\leq C \sum_{Q \in \mathcal{Q}(\rho)} H^d(\overline{X} \cap Q \setminus \mathcal{S}_d) \\ &\leq C' H^d(\overline{X} \setminus \mathcal{S}_d), \end{aligned}$$

by (5.27) and because the cubes Q have bounded overlap. Altogether,

$$(5.59) \quad \begin{aligned} H^d(\Phi(F \cap W)) &= H^d(f(X)) \\ &\leq H^d(f(X_1)) + H^d(f(X_2)) \\ &\leq H^d(X_1) + C' H^d(\overline{X} \setminus \mathcal{S}_d) \end{aligned}$$

by (5.54), (5.21), and (5.58). Next we check that

$$(5.60) \quad \mathcal{H} \dim(\overline{X} \setminus [X \cup \mathcal{S}_d]) \leq d - 1.$$

Indeed let $z \in \overline{X} \setminus X$ be given. Recall from (5.21) that every point of X is of the form $h(\psi(y))$ for some $y \in F \cap W$. Since $W \subseteq B_2$ and the domain of definition V_+ of h is closed, we can also write $z = h(\psi(y))$, but with $y \in F \cap \overline{W}$. In addition, $y \notin W$, because $z \notin X$. So $\psi(y) = y$. But $y \in F \subset V$ by (5.17), so $h(y) = y$ by (5.11). Altogether $z = h(\psi(y)) = y \in F$. If $z \in F^*$, then it lies on \mathcal{S}_d . So $\overline{X} \setminus [X \cup \mathcal{S}_d] \subset F \setminus F^*$, and (5.60) follows from (4.13).

Because of (5.60), the right-hand side of (5.59) is at most $H^d(X_1) + C'H^d(X \setminus \mathcal{S}_d) = H^d(X_1) + C'H^d(X_2) \leq C'H^d(X)$. Altogether,

$$\begin{aligned}
 H^d(F \cap W) &\leq MH^d(\Phi(F \cap W)) \\
 &\leq C'MH^d(X) \\
 (5.61) \quad &= C'MH^d(h \circ \psi(F \cap W)) \\
 &\leq 4^{nd}C'MH^d(\psi(F \cap W))
 \end{aligned}$$

by (5.56), (5.59), (5.21), and (5.57).

Recall from (5.2) that $S = F \cap B_2$ and from (5.6) that $W \subset B_2$. So $S \cap W = F \cap W$, and (5.61) is just the same as (5.7).

This completes our proof of quasiminimality for F . Thus (5.2) holds (with $k = 4^{nd}C'M$), and we shall be able to apply the results of [DS3].

6. Local uniform rectifiability and flatness

In this section we use the local uniform rectifiability of F^* that comes from (5.2) to show that F^* is often flat, derive a contradiction with our assumption (1.2), and thus complete the proof of Lemma 4.1.

Let us first apply the main result of [DS3], which is stated on pp. 10 and 11. Call S^* the closed support in B_2 of the restriction of H^d to S , as in (1.12) on [DS3, p. 9]. That is,

$$(6.1) \quad S^* = \{x \in B_2; H^d(S \cap B(x, r)) > 0 \text{ for all } r > 0\}.$$

Recall from (5.2) that $S = F \cap B_2$. Since F is a finite union F^* of d -faces, plus a set of Hausdorff dimension at most $d - 1$ (see the definition of \mathcal{F}_0 below (4.12)), we get that

$$(6.2) \quad S^* = F^* \cap B_2.$$

Theorem 2.11 in [DS3] says that S^* is locally Ahlfors-regular and uniformly rectifiable in B_2 . More precisely, for each $y \in S^* \cap B_1$, there is a closed, Ahlfors-regular and uniformly rectifiable set Z_y of dimension d such that

$$(6.3) \quad S^* \cap B(y, \delta/2) \subset Z_y \subset S^* \cap B(y, \delta).$$

Recall from (4.4) that B_1 is a ball a little bit smaller than B_2 , and from (5.3) and (4.6) that $\delta = 4^{-n-1}L \leq \eta r_1/100$. In the statement of [DS3] one allows all balls $B(y, R)$ centered on S^* , with radius $R < \delta$, and such that $B(y, 3R) \subset B_2$, but we shall only need the ones above.

We should also put stress on the fact that the Ahlfors-regularity and uniform rectifiability constants for the Z_y depend on our various constants, but not on ρ or α_2 . The convention in this section will be that C denotes a constant that depends on $n, d, \alpha, \eta, \delta_1$, and even C_1 , but in particular not on ρ or α_2 .

Let us not define uniform rectifiability here, and refer to [DS1] for all relevant information, but recall that Ahlfors-regular means that there is a constant C such that

$$(6.4) \quad C^{-1}t^d \leq H^d(Z_y \cap B(z, t)) \leq Ct^d \text{ for } z \in Z_y \text{ and } 0 < t \leq \text{diam}(Z_y).$$

In the present case, we also have that

$$(6.5) \quad \delta/4 \leq \text{diam}(Z_y) \leq 2\delta.$$

This is Remark 2.15 in [DS3]; grosso modo, the point is that if $\text{diam}(S^* \cap B(y, R)) < \delta/4$ we could contract $S^* \cap B(y, R)$ into a point, and this would contradict the quasiminimality of S .

Cover $S \cap B_1$ with less than C balls $B(y, \delta/2)$ centered on $S^* \cap B_1$, and call Z the corresponding union of sets Z_y . Then Z is a uniformly rectifiable Ahlfors-regular set, of diameter comparable to δ and r_1 , and

$$(6.6) \quad S^* \cap B_1 \subset Z.$$

[For the reader that would be worried about taking (finite) unions of uniformly rectifiable sets, let us say that this is just a matter of convenience, and we could have worked with the various Z_y too.]

We shall not work with uniform rectifiability directly, but with one of its consequences, the “weak geometric lemma”. Set

$$(6.7) \quad \beta_Z(y, t) = \inf_{P \in \mathcal{P}} \left\{ \sup \{ t^{-1} \text{dist}(z, P); z \in Z \cap B(y, t) \} \right\}$$

for $y \in Z$ and $t > 0$, where \mathcal{P} still denotes the set of affine d -planes. This is the same thing as β_E in (1.2), but relative to Z . Set

$$(6.8) \quad \varepsilon = \frac{\varepsilon_0}{10},$$

where ε_0 is still the constant in (1.2), and consider the bad set

$$(6.9) \quad \mathcal{B}(\varepsilon) = \{(y, t); y \in Z, 0 < t < \delta, \text{ and } \beta_Z(y, t) > \varepsilon\}.$$

The weak geometric lemma says that there is a constant $C(\varepsilon) > 1$, that depends only on ε and the Ahlfors-regularity and uniform rectifiability constants for Z , such that

$$(6.10) \quad \int_{y \in Z \cap B(y_0, R)} \int_{0 < t < R} \mathbf{1}_{\mathcal{B}(\varepsilon)} \frac{dH^d(y) dt}{t} \leq C(\varepsilon) R^d$$

for $y_0 \in Z$ and $0 < R < \text{diam}(Z)$. The simplest reference for this is probably the following. Theorem 2.4 on [DS1, p. 32] says that uniform rectifiability implies the bilateral weak geometric lemma (BWGL). The BWGL is defined on [DS1, p. 32], and the reader may easily check that it is stronger than the weak geometric lemma (or (6.10)). Let us also say that since [DS1] prefers to talk about unbounded Ahlfors-regular sets Z , we may need to apply their Theorem 2.4 to $Z' = Z \cup P$, where P is some d -plane in \mathbb{R}^n such that $\text{dist}(P, B_1) \leq r_1$. It is very easy to check that Z' is uniformly rectifiable (once you are given the definition), and the weak geometric lemma for Z' clearly implies (6.10).

So we have (6.10). It will be convenient to set

$$(6.11) \quad d\mu = \frac{dH^d(y) dt}{t}.$$

Note that because of $\frac{dt}{t}$, μ is locally infinite (and essentially scale invariant).

Actually, we shall only use (6.10) at one scale. That is, we shall only need to know that

$$(6.12) \quad \mu(\mathcal{B}(\varepsilon)) = \iint_{(y,t) \in \mathcal{B}(\varepsilon)} \frac{dH^d(y) dt}{t} \leq C(\varepsilon) r_1^d,$$

which is an easy consequence of (6.10). [If you are worried about the measure of the part of $\mathcal{B}(\varepsilon)$ where $\text{diam}(Z) < t < \delta$, note that $\int_{y \in Z} \int_{10^{-2}\delta \leq t \leq \delta} d\mu \leq 10H^d(Z) \leq Cr_1^d$.]

Since μ is infinite, (6.12) really says that $\mathcal{B}(\varepsilon)$ is small, or equivalently that $\beta_Z(y, t) \leq \varepsilon$ most of the time. The general idea now to use this and the fact that

$$(6.13) \quad \begin{aligned} H^d(Z \cap E_\rho \cap B_0) &\geq H^d(S^* \cap E_\rho \cap B_0) \\ &= H^d(F^* \cap E_\rho \cap B_0) \\ &= H^d(F \cap E_\rho \cap B_0) \geq \delta_1 r_1^d / 2, \end{aligned}$$

by (6.6), (6.2), (4.13), and (4.32), to find some place where E_ρ is very flat and get a contradiction. Since apparently we cannot use density

results to find directly large balls where E_ρ and S^* essentially coincide, we shall have to have resort to a slightly more complicated accounting argument.

Let C_2 be a large constant, to be chosen soon; C_2 will depend on many things, including ε_0 , but this is still all right. Set

$$(6.14) \quad \mathcal{G}_0 = \{(y, t); y \in Z \cap E_\rho \cap B_0 \text{ and } C_2\rho \leq t < \delta\},$$

and then

$$(6.15) \quad \mathcal{G} = \{(y, t) \in \mathcal{G}_0; \beta_Z(y, t) \leq \varepsilon\}.$$

To check that \mathcal{G} is reasonably large, first note that

$$(6.16) \quad \mu(\mathcal{G}_0) = H^d(Z \cap E_\rho \cap B_0) \int_{C_2\rho}^{\delta} \frac{dt}{t} \geq \frac{\delta_1 r_1^d}{2} \text{Log} \left(\frac{\delta}{C_2\rho} \right),$$

by (6.13). Later on, we shall take α_2 so small (in (4.3)) that this will be as big as we want. But for the moment, let us just record that $\mathcal{G} = \mathcal{G}_0 \setminus \mathcal{B}(\varepsilon)$, so

$$(6.17) \quad \mu(\mathcal{G}) \geq \mu(\mathcal{G}_0) - \mu(\mathcal{B}(\varepsilon)) \geq \frac{\delta_1 r_1^d}{2} \text{Log} \left(\frac{\delta}{C_2\rho} \right) - C(\varepsilon)r_1^d,$$

by (6.12). Next we want to give an upper bound for $\mu(\mathcal{G})$ that will contradict (6.17).

Our plan is to associate a reasonably large piece of E_ρ to each pair $(y, t) \in \mathcal{G}$, with reasonable overlap, and then deduce our upper bound from (3.4).

So let $(y, t) \in \mathcal{G}$ be given. Since $\beta_Z(y, t) \leq \varepsilon$, we can find a d -plane P such that

$$(6.18) \quad \text{dist}(z, P) \leq \varepsilon t \text{ for } z \in Z \cap B(y, t).$$

Besides, $y \in E_\rho$ by definition of \mathcal{G}_0 . Thus we can find $\xi \in E$ such that

$$(6.19) \quad |\xi - y| \leq \sqrt{n}\rho \leq C_2\rho/10 \leq t/10,$$

by (3.2), if C_2 is large enough, and because $(y, t) \in \mathcal{G}_0$. By (1.2), $\beta_E(\xi, t/2) \geq \varepsilon_0$, and so we can find $\zeta \in E \cap B(\xi, t/2)$ such that

$$(6.20) \quad \text{dist}(\zeta, P) \geq \frac{\varepsilon_0 t}{3} = \frac{10\varepsilon t}{3}.$$

Set

$$(6.21) \quad D(y, t) = E_\rho \cap B(\zeta, \varepsilon t).$$

Note that $|\zeta - y| \leq |\xi - y| + t/2 \leq t/10 + t/2 = 6t/10$ (by (6.19)), so

$$(6.22) \quad D(y, t) \subset E_\rho \cap B(y, t).$$

Also, $\text{dist}(\zeta, Z \cap B(y, t)) \geq \text{dist}(\zeta, P) - \varepsilon t \geq \frac{10\varepsilon t}{3} - \varepsilon \geq 2\varepsilon t$ by (6.18) and (6.20), and $\text{dist}(\zeta, Z \setminus B(y, t)) \geq 4t/10$ because $|\zeta - y| \leq 6t/10$. Altogether, $\text{dist}(\zeta, Z) \geq 2\varepsilon t$ and

$$(6.23) \quad \text{dist}(D(y, t), Z) \geq \varepsilon t.$$

These estimates will help with the bounded overlap. But first we want to show that $D(y, t)$ is not too small. We claim that

$$(6.24) \quad H^d(D(y, t)) = H^d(E_\rho \cap B(\zeta, \varepsilon t)) \geq \delta_1(\varepsilon t)^d.$$

The first part comes from (6.21). For the second part, let us try to apply *TND* to the ball $B(\zeta, \varepsilon t)$. First we check (3.3) with $x_1 = \zeta$ and $r_1 = \varepsilon t$. Clearly $\zeta \in E$, $\varepsilon t \leq \delta \leq r_0$, and $\rho \leq C_2^{-1}t \leq (2\sqrt{n})^{-1}\alpha\varepsilon t$ because $(y, t) \in \mathcal{G}_0$ and if C_2 is large enough. If the analogue of (3.4) does not hold, i.e., if $H^d(E_\rho \cap B(\zeta, \varepsilon t)) > C_1(\varepsilon t)^d$, we are happy because (6.24) is satisfied (with a much better constant). Otherwise, we get a smaller ball $B \subset B(\zeta, \varepsilon t)$ where (3.6) holds for all appropriate families $\{\varphi_t\}$ of deformations. We take $\varphi_t(y) = y$ everywhere, which satisfies the required conditions trivially, and now (3.6) says that $H^d(E_\rho \cap B(\zeta, \varepsilon t)) \geq H^d(E_\rho \cap B) \geq \delta_1(\varepsilon t)^d$. So (6.24) holds in all cases.

We are now ready for the accounting. In the following computations, we assume that all functions are measurable. One could avoid this difficulty by taking exterior measures, or by discretizing the computations, but we prefer to ignore the issue. First write that

$$(6.25) \quad \mu(\mathcal{G}) = \iint_{(y,t) \in \mathcal{G}} \frac{dH^d(y) dt}{t} \leq C \iiint_{(y,t) \in \mathcal{G}; w \in D(y,t)} \frac{dH^d(w) dH^d(y) dt}{t^{d+1}}$$

by (6.11) and (6.24), and where we do not write the dependence on ε either. Next we want to exchange the order of integration. First observe that $w \in E_\rho \cap B(y, t) \subset E_\rho \cap B(x, r)$ by (6.22), because $y \in B_0$ by (6.14), and because $t \leq \delta < \eta r_1/100$ by (6.14), (5.3), and (4.6). Thus

$$(6.26) \quad \mu(\mathcal{G}) \leq C \int_{w \in E_\rho \cap B(x, r)} I(w) dH^d(w)$$

where the inside integral is

$$(6.27) \quad I(w) = \int \int_{(y,t) \in \mathcal{G}; w \in D(y,t)} \frac{dH^d(y) dt}{t^{d+1}}.$$

Now fix $w \in E_\rho \cap B(x, r)$ and $t \leq \delta$, and integrate in y first. Note that $y \in Z \cap B(w, t)$, by (6.14) and (6.22). By local Ahlfors-regularity of Z

(see (6.4)), $H^d(Z \cap B(w, t)) \leq Ct^d$, and when we integrate again in t we get that

$$(6.28) \quad I(w) \leq \int_{t \in T(w)} H^d(Z \cap B(w, t)) \frac{dt}{t^{d+1}} \leq C \int_{t \in T(w)} \frac{dt}{t},$$

where now we integrate over the set $T(w)$ of radii t for which we can find y such that $w \in D(y, t)$. Note that if $w \in D(y, t)$, then $\text{dist}(w, Z) \geq \text{dist}(D(y, t), Z) \geq \varepsilon t$ by (6.23), and on the other hand $\text{dist}(w, Z) \leq |w - y| \leq t$. So $T(w) \subset [\text{dist}(w, Z), \varepsilon^{-1} \text{dist}(w, Z)]$, $I(w) \leq C$, and (6.26) yields

$$(6.29) \quad \mu(\mathcal{G}) \leq CH^d(E_\rho \cap B(x, r)) \leq CC_1 r_1^d,$$

because $B(x, r) \subset B(x_1, r_1)$ and we assumed that (4.2) fails.

When we compare this with (6.17), we get that

$$(6.30) \quad \frac{\delta_1 r_1^d}{2} \text{Log} \left(\frac{\delta}{C_2 \rho} \right) - C(\varepsilon) r_1^d \leq CC_1 r_1^d.$$

Maybe we should recall here that our constants δ_1 , δ (defined by (5.3) and (4.6)), and C_2 (see near (6.14)) are now chosen, and that we still have the right to take α_2 (in the statement of Lemma 4.1) as small as we wish. See the comments before (4.3). Also recall from (4.3) that $\rho \leq \alpha_2 r_1$; if we choose α_2 small enough, ρ is as small as we want compared to δ/C_2 , and (6.30) fails.

We finally reached the contradiction that proves Lemma 4.1. \square

7. A lower bound on Hausdorff dimension

In this section we deduce Theorem 3.8 from Lemma 4.1. So we continue with the same closed set E as in the statement of Theorem 3.8. First we want to choose a constant C_1 , so that we can apply the lemma. We give the definition now so that the reader will be convinced that there is no vicious circle, but it will look a little weird.

For each $\rho > 0$ and each $y \in \mathbb{R}^n$, call $R_\rho(y)$ the union of the skeletons $S_d(Q)$, where $Q \in \mathcal{Q}(\rho)$ meets $\overline{B}(y, 10\rho)$. Clearly $H^d(R_\rho(y)) \leq A\rho^d$ for some geometric constant $A = A(n, d)$; we choose $C_1 = 10A$.

Now we can apply Lemma 4.1. This gives constants r_0 and α_2 such that

$$(7.1) \quad H^d(E_\rho \cap B(x, r)) > C_1 r^d$$

for all choices of $x \in E$, $r \in (0, r_0)$, and $\rho \in (0, \alpha_2 r]$. Choose an integer N such that $\frac{1}{2}\alpha_2 \leq N < \alpha_2$; this will make our computations look nicer.

For $x \in E$ and $r \in (0, r_0)$, set $\rho = N^{-1}r$ and pick a maximal collection $H(x, r)$ of points of $E \cap B(x, 2r)$, subject to the constraint that

$$(7.2) \quad |y - z| \geq 10\rho \text{ for } y, z \in H(x, r), y \neq z.$$

We want to show that

$$(7.3) \quad \#H(x, r) \geq 10N^d.$$

Let us check first that

$$(7.4) \quad \text{the sets } R_\rho(y), y \in H(x, r), \text{ cover } E \cap B(x, r).$$

Let $z \in E \cap B(x, r)$ be given. Then $z \in S_d(Q)$ for some $Q \in \mathcal{Q}(\rho)$ that meets E . Call y_1 a point of $E \cap Q$. By maximality of $H(x, r)$, we can find $y \in E$ such that $|y - y_1| \leq 10\rho$. Then Q is one of the cubes that meet $\overline{B}(y, 10\rho)$, and $S_d(Q) \subset R_\rho(y)$. This proves (7.4).

Now

$$H^d(E \cap B(x, r)) \leq \sum_{H(x, r)} H^d(R_\rho(y)) \leq A\rho^d(\#H(x, r)) = AN^{-d}r^d(\#H(x, r)),$$

and so

$$\#H(x, r) \geq A^{-1}N^dr^{-d}H^d(E \cap B(x, r)) \geq A^{-1}N^dC_1 = 10N^d,$$

by (7.1).

The simplest way to estimate the dimension of E will be to use (7.4) to embed a Cantor set $T^{\mathbb{N}^*}$ in E , where $T = \{1, 2, \dots, 10N^d\}$ is a set with $10N^d$ elements. Set

$$(7.5) \quad \rho_k = N^{-k}r_0/2 \text{ for } k \geq 0.$$

We want to define a point $x_j \in E$ for every choice of $j \in T^k$ and $k \geq 0$.

We start with any point $x \in E$, which we call x_\emptyset to start the induction. Then assume that we already constructed the points x_j , $j \in T^k$, and let us do this for $j \in T^{k+1}$.

Let $j \in T^k$ be given, and apply (7.3) to the point x_j and the radius ρ_k . We get a set $H(x_j, \rho_k)$ of points of $E \cap B(x_j, 2\rho_k)$, that lie at mutual distances at least $10\rho_{k+1}$ from each other. Pick $10N^d$ points in this set, and call them $x_{j,1}, x_{j,2}, \dots, x_{j,10N^d}$. Here we use the convenient notation (j, ℓ) for points of T^{k+1} that start with j .

This completes our definition of x_j , $j \in T^k$, by induction on k . Note that in the construction above, $|x_{j,\ell} - x_j| \leq 2\rho_k$. An easy induction gives that

$$(7.6) \quad |x_{j'} - x_j| \leq 3\rho_k \text{ when } j \in T^k, j' \in T^{k'}, k' \geq k, \text{ and } j' \text{ starts with } j.$$

Similarly, recall that the points $x_{j,\ell}$ lie at distances at least $10\rho_{k+1}$ from each other. Since the further generations will not modify this by more than $6\rho_{k+2} < \rho_{k+1}$, by (7.6), we get that

$$(7.7) \quad |x_{j'} - x_j| \geq 9\rho_k \text{ when the } k\text{-th components of } j \text{ and } j' \text{ differ.}$$

Now we put a distance on $K = T^{\mathbb{N}^*}$. If j and j' are two distinct sequences in K , call $k(j, j')$ the smallest integer k such that $j_k \neq j'_k$, and then set

$$(7.8) \quad \text{dist}(j, j') = \rho_{k(j, j')} = N^{-k(j, j')} r_0 / 2.$$

This defines a distance on K , for which K is closed.

By (7.6), we can define $h: K \rightarrow E$ by $h(j) = \lim_{k \rightarrow +\infty} x_{j_1, j_2, \dots, j_k}$, where we write $j = (j_1, j_2, \dots, j_k, \dots)$. Moreover, if we apply (7.6) with $k = k(j, j') - 1$, we get that $|h(j) - h(j')| \leq 3\rho_{k(j, j')-1} = 3N \text{dist}(j, j')$.

Similarly, (7.7) and a small limiting argument say that $|h(j) - h(j')| \geq 9\rho_{k(j, j')} = 9 \text{dist}(j, j')$. Altogether, h is biLipschitz, and

$$(7.9) \quad \mathcal{H} \dim(E) \geq \mathcal{H} \dim(K).$$

Set $d^* = \mathcal{H} \dim(K)$. We claim that

$$(7.10) \quad d^* = d + \frac{\text{Log } 10}{\text{Log } N} > 1.$$

Indeed it is known that d^* is given by the equation

$$(7.11) \quad (10N^d)^{-k} = N^{-kd^*},$$

where the left-hand side gives the measure (for the obvious uniform measure on K) of a box of the k -th generation, while the right-hand side is an equivalent of the d^* -th power of the diameter of the box. See [Fa] for a proof of (7.11) and similar computations on self-similar sets.

When we take the Logarithm of both sides of (7.11) and divide by $-k$, we get $\text{Log } 10 + d \text{Log } N = d^* \text{Log } N$, which is equivalent to (7.10). Thus $\mathcal{H} \dim(E) \geq d^*$, by (7.9).

The constant N , and hence also d^* , depends only on the various constants in the construction, as promised after the statement of the theorem. This completes our proof.

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